Economic foundations of generalized games with shared constraint: Do binding agreements lead to less Nash equilibria?

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July 29, 2022

Abstract

A generalized game is a situation in which interaction between agents occurs not only through their objective function but also through their strategy sets; the strategy set of each agent depends upon the decision of the other agents and is called the individual constraint. As opposed to generalized games with exogenous shared constraint literature pioneered by [Rosen, 1965], we take the *individual constraints as the basic premises* and derive the shared constraint generated from the individual ones, a set K. For a profile of strategies to be a Nash equilibrium of the game with individual constraints, it must lie in K. But if, given what the others do, each agent agrees to restrict her choice in K, something that we call an endogenous shared constraint, this mutual restraint may generate new Nash equilibria. We show that the set of Nash equilibria in endogenous shared constraint contains the set of Nash equilibria in individual constraints. In particular, when there is no Nash equilibrium in individual constraints, there may still exist a Nash equilibrium in endogenous shared constraint. We also prove a few results for a specific class of generalized games that we call non-classical games. Finally, we give two economic applications of our results to collective action problems: carbon emission and public good problems.

Keywords: Game theory, generalized games, binding agreements, individual and shared constraints, collective action problems

JEL Classification: C72, D7, F53

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1 Introduction

It has long been recognized in Economics, Political sciences, International relations and more generally in Social sciences that Game theory is a useful tool to understand (and/or to predict) the outcome of a particular economic or social interaction between a set of agents (see [Fudenberg and Tirole, 1991], [Brams, 2011], [Schelling, 1980], [Moulin, 1986] for classical textbooks). A striking feature of most (not to say all) applied game theoretical models is that interaction between agents only takes place through the objective functions (utility or cost function). Given what the others choose, the aim of a given agent is to optimize her objective function by choosing the optimal strategy in a given set assumed to be invariant with respect to the choice of the other agents.

When each agent explicitly faces for instance a common binding constraint, the decision of a given agent may not only impact the objective function of the other agents but also their strategy set. Consider the well-known example of international environmental agreements (such as the Kyoto protocol) in which the total volume of emissions of greenhouse gas must be lower than a given threshold $\overline{e}$. From the point of view of a given country i, given the sum of emissions of greenhouse gas of the other countries e_{-i}, its total emission e_i explicitly depends upon the emissions of the other agents since the strategy set of country i is equal to $S_i(e_{-i}) = [0, \overline{e} - e_{-i}]$. Within such a simple and natural framework with a collective binding constraint, interaction between agents may not only take place through the objective function but also through their strategy set (see [Breton et al., 2006] and [Tidball and Zaccour, 2005] for early economic applications). Games in which the interaction takes place not only through the utility function (or cost function) but also through the strategy sets are called generalized games or Generalized Nash Equilibrium Problem (GNEP for short) (see [Facchinei and Kanzow, 2007], [Fischer et al., 2014]). Throughout this paper, we may call interchangeably GNEPs and generalized games.

Generalized games (GNEPs) are not recent and have received considerable attention in operational research. For instance, in their well-known survey, [Facchinei and Kanzow, 2007] offer a historical overview of GNEPs dating back to the seminal papers of [Arrow and Debreu, 1954], [Nash, 1950], [Nash, 1951], and they provide interesting examples of applications of such games in telecommunication or in environmental pollution. In the applied maths literature more generally, there has been an abundant number of articles on GNEPs in recent years either proposing new methods, existence results or numerical algorithms to find a Nash equilibrium (e.g., [Facchinei et al., 2009], [Aussel and Dutta, 2008], [Fischer et al., 2014]). However, to the best of our knowledge, there has been only few papers trying to apply GNEPs in Economics (but see [Breton et al., 2006], [Elfoutayeni et al., 2012], [Le Cadre et al., 2020]).

A generalized game with *individual constraints* can be naturally defined as a game in which each agent has to satisfy her own individual constraint, that is, each agent i is required to pick a strategy x_i in a set which is (possibly) dependent on the strategies picked by the other agents. In this sense, classical games but also GNEPs can be seen as generalized games with individual constraints.

A generalized game with *shared constraint* is a specific category of games with individual constraints and has been introduced for the first time in [Rosen, 1965] although the term shared constraint does not appear[1]. In his seminal paper, [Rosen, 1965] pioneered such games and proves an existence result about concave games. Remarkably, Rosen's result is true not only on E defined as the classical Cartesian product of strategy sets but also on any closed and bounded convex subset $X \subset E$. Later on in the literature, X has been called the *shared constraint set.* Such a game is called generalized game with shared constraint in the sense that all the agents share the same constraint, that is, the profile of strategies $x := (x_1, ..., x_n)$ must always remain in the shared constraint set X: given x_{-i}, each agent i is required to pick a strategy $x_i \in E_i$ such that $x := (x_i, x_{-i}) \in X$.

The striking feature of the shared constraint approach developed in [Rosen, 1965] and the subsequent literature is that this shared constraint set X is *exogenously given* and bears no relationship with any possible individual constraints. Later on, [Bensoussan, 1974] introduced a variational formulation of the equilibrium of a generalized game with shared constraint and the literature on GNEPs now formulate the equilibrium as a (quasi) variational inequality (see also [Harker, 1984] and [Harker, 1991]). In its most basic version (see [Fischer et al., 2014] or [Facchinei and Kanzow, 2007] for excellent review papers), the variational formulation involves the partial derivative of the objective function of each agent with respect to her own strategy, which means that the underlying functions (objective function/constraints function) have to be differentiable.

It is the aim of this methodological paper to show the fruitfulness of such generalized games with shared constraints in the Economics of binding constraints[2]. We show how, by taking an alternative road-map to the traditional one encountered in the (mathematical) literature on GNEPs with shared constraints, our new methodological approach can be used to provide some foundations to strategic collective actions problems in Economics.

We re-consider in this paper GNEPs with shared constraints and our aim is to shed light on the relationships between the individual constraints on the one hand and the shared constraints on the other hand. Instead of considering a shared constraint set which is *exogenously given* and imposed to the set of agents, as in the literature on GNEPs, we consider these individual constraints as the basic premises of the game and derive an *endogenous shared constraint set* generated precisely by these individual constraints. From a pure economic point of view, the existing literature on generalized games with shared constraints has two limitations. First, as said, the shared constraint is in general postulated and not generated from the individual constraints. Second, by formulating the equilibrium of the game as a variational inequality, the characterization of the equilibrium excludes the simplest games in which the strategy set of each agent is a finite set (e.g., the 2-2 games).

We prove in this paper the following simple result for which the economic applications are

[1]Historically, generalized games were first introduced by [Debreu, 1952]. In [Harker, 1991] or in [Krawczyk, 2007], the authors note that a number of different names appeared in the literature to define these generalized games, abstract economy, social equilibria games, pseudo-Nash equilibria games or normalized equilibria ([Rosen, 1965]).

[2]It may be the result of an endogenous cooperation between the agents or the result of some constraining regulation.

numerous: the set of Nash equilibria of a generalized game with individual constraints is *included* in the set of Nash equilibria of the generalized game with shared constraint generated from these individual constraints. Interestingly, [Feinstein and Rudloff, 2021] proved independently a similar result (see Theorem 3.2. in their paper) within a more abstract framework. This result, whose proof turns out to be simple, has two basic important consequences.

1. There are situations in which there is no Nash equilibrium in a game with individual constraints while such Nash equilibria exist in the game with shared constraint generated from the individual ones.

2. If there is no Nash equilibrium in shared constraint, then, no Nash equilibrium in the game with individual constraints can exist (the converse is however not true).

From an economic point of view, our result requires some binding agreements or a a constraining regulation; given what the others do, each agent i agrees or is constrained to pick a strategy not in her basic set of strategy (individual constraint) but in the shared constraint set that results from these individual constraints. Within our generalized game, sharing the constraint means in general that, given what the others do, compared with the primitive individual constraints X_i, each agent will now have to choose a strategy in a *smaller set*, that is, in $K_i \subset X_i$. Some strategies that were available in the game with individual (primitive) constraints are not anymore available in the game with shared constraint. Put it differently, introducing a shared constraint is equivalent to introducing restrictions and this kind of restriction exactly fits the notion of mutual restraint discussed in the well-known book of [Barrett, 2007] entitled *Why cooperate?*.

It is clear that a binding agreement requires some form of cooperation between the agents. In his well-known textbook, [Moulin, 1995] considers three modes of cooperation between a set of agents, direct agreements, decentralized behaviour and justice. In this paper, while not explicitly modeled, the mode of cooperation considered is the first one, direct agreements, and can be thought (from a game theoretical point of view) of as the result of preplay communication[3]. These direct binding agreements are particularly important when there is no Nash equilibrium in the game with individual constraints while (at least) one equilibrium exists in the game with shared constraint. We illustrate this idea in the second part of the paper to collective action problems, an externality-pollution and a public good problem. In each model, we show situation in which there is no Nash equilibrium in individual constraints while there may be (at least) one Nash equilibrium in shared constraint.

The remainder of the paper is structured as follows. In Section 2, we remind the definitions of generalized games as well as the corresponding notions of Nash equilibria. We then define the notion of generalized game with shared constraint generated from a game with individual constraints,

[3]As observed in [Moulin, 1995], transaction cost is a drawback to this direct agreement mode. If this preplay communication is long and difficult, the transaction cost will be high. This problem of commitment is also discussed in the well-known book of [Ostrom, 1990]

something we call endogenous shared constraint. In section 3, to present our concept of endogenous shared constraint, we offer a gallery of 2×2 games, along the lines of [Fishburn and Kilgour, 1990], [Kilgour and Fraser, 1988]. In section 4, we prove that the set of Nash equilibria of a generalized game with individual constraints is included in the set of Nash equilibria of the generalized game with endogenous shared constraint. Moreover, we prove additional results regarding a specific class of generalized games called non-classical games in which interaction between agents occur only through the constraints. In Section 5, we offer two different collective action problem models which illuminate our results when no Nash equilibrium exists in individual constraints. The first model is a public good problem while the second one is an environment control problem and we show the economic usefulness of the introduction of an endogenous shared constraint in these problems. Section 6 concludes the paper.

2 Games with individual and shared constraints

2.1 Generalized games with individual constraints

We consider a game with $N \geq 2$ agents (or players) and we denote $J = \{1, ..., N\}$ the set of players. The decision (or control) variable of each player $i \in J$ is denoted by $x_i \in E_i$, where E_i is a subset of $\mathbb{R}^{n_i}$ called the strategy set. Let $E = \prod_{i=1}^{N} E_i = E_1 \times \ldots \times E_N$ and denote by $x \in E$ the vector formed by all these decision variables (strategies) which has dimension $n := \sum_{i=1}^{N} n_i$ so that $E \subset \mathbb{R}^n$. As usual in game theory, we denote by $x_{-i} \in E_{-i}$ the vector formed by all the players' decision variables except those of player i. To emphasize the i-the player's strategy, we sometimes write $(x_i, x_{-i}) \in E_i \times E_{-i}$ instead of $x \in E$. Each player i has an objective function $\theta_i : \mathbb{R}^n \to \mathbb{R}$ that depends on both his own decision variables x_i as well as on the decision variables x_{-i} of all other players. We denote the objective function of player i by $\theta_i(x_i, x_{-i})$. For a given $x_{-i} \in E_{-i}$ and depending upon the game, the aim of agent i may be to maximize or to minimize its objective function $\theta_i(x_i, x_{-i})$. In general, when this objective function is a utility (or a payoff) function, it is the aim of the agent to maximize it while when it is a cost (or a loss) function, it is the aim of the agent to minimize it. Throughout the article, unless otherwise specified, we will assume that the objective function is a cost function so that each agent i, given the other players' strategies x_{-i}, is seeking a strategy x_i to minimize $\theta_i(x_i, x_{-i}) = \theta_i(x)$. Throughout the paper, we may use interchangeably the terms decision, decision variable and strategy and we only consider the case of pure strategy, that is, the situation in which each agent chooses a strategy with probability one.

In a generalized game, each player $i \in J$ must pick a strategy $x_i \in X_i(x_{-i}) \subseteq E_i$ where the set $X_i(x_{-i})$ explicitly depends upon the rival players' strategies. As in classical games in which the strategy set E_i of each agent i is given, in generalized games, the strategy set that we call the individual constraint function $X_i(x_{-i})$ (or simply the individual constraint) is also exogenously given. To define in full generality the individual constraint which depends upon the decision of others, let X_i be defined as follows:

$$X_i : E_{-i} \to \mathcal{P}(E_i) \quad i = 1, 2, ..., N \tag{1}$$

where $\mathcal{P}(E_i)$ is the power set of E_i. It is usual to call X_i a point-to-set map (ot a set-valued map) since it associates a subset of E_i to each point of E_{-i}. This means that for a given point $x_{-i} \in E_{-i}$, the strategy set also called the individual constraint of agent i is equal to $X_i(x_{-i}) \subseteq E_i$, a subset of E_i.

We want to focus on the economic foundation of generalized games. For instance, in a strategic interaction with two agents, given the choice x_2 of agent 2, agent 1 may have to choose x_1 in E_1 subject to a constraint of the form $g_1(x_1, x_2) = x_1 + bx_2 \leq r_1$. Put it differently, the strategy set of agent 1 explicitly depends upon the choice of agent 2. Let us consider two economic examples of such a situation that will be considered in detail in section 5.

- In an environmental problem with externalities, agent 1 (i.e., country 1) may be constrained to choose its emission of greenhouse gas x_1 subject to a constraint of the form $g_1(x_1, x_2) = x_1 + b_2x_2 \leq r_1$ where r_1 is the maximum emission of country 1 and b_2x_2 is the impact emission of country 2 on country 1.

- In a public good problem, following [Guttman, 1978], agent 1 has to choose a flat contribution x_1 plus a matching rate bx_2 (which is a function of the flat contribution of agent 2 where $b \in (0, 1]$) so that $g_1(x_1, x_2) = x_1 + bx_2 \leq r_1$ is the budget constraint of agent 1.

When thinking about collective action problems, there are various situations in which the decision of a given agent has an impact on the constraint of another agent.

Definition 1 *The 4-uplet* $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ *is called a generalized game with individual constraints.*

It should be clear "classical games" encountered in Economics appear as a particular generalized games with individual constraints. When, for each $i \in J$, X_i is invariant with respect to the choice of the other agents, we are back to a classical game in which the interaction only takes place through the objective functions. In such a case, X_i reduces to E_i.

2.2 Generalized games with endogenous and exogenous shared constraint: definitions and economic foundations

For what we shall call conventional or classical games, that is, when for each i, E_i is invariant with respect to x_{-i}, recall that the best response BR_i of an agent i is defined as

$$\begin{array}{rcll} BR_i & : & E_{-i} & \to & E_i \\ & & x_{-i} & \mapsto & \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \arg\min_{x_i \in E_i} \theta_i(x_i, x_{-i})\} \end{array}$$

that is, given a vector of strategies of other players x_{-i}, the best response gives the set of optimal strategie(s) for agent i. Unless mentioned otherwise, BR_i needs not be a singleton and is indeed in general a point-to-set map. As we shall now see, when studying a generalized game, different constraints will inevitably lead to different best response functions.

Some definitions. In a generalized game, as opposed to classical games, it may be the case that for some x_{-i}, the strategy set of agent i is simply empty, that is, $X_i(x_{-i}) = \emptyset$, which means that the objective function is undefined. As a result, the best response is also undefined so that the equilibrium can not exist. Such an empty set problem never occurs in classical games since the strategy set of each agent i is invariant with respect to x_{-i}. When $x \in E$ is such that $x_i \in X_i(x_{-i})$ for each agent $i \in J$, we say that the profile of strategies x is *admissible.*

Definition 2 *For a given generalized game with individual constraints* $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$*, let* K *be the subset of* E *defined as follows.*

$$K = \{x \in E, \ \forall i \in J, \ x_i \in X_i(x_{-i})\} \tag{2}$$

K *is called the set of admissible strategies of the generalized game with individual constraints.*

The set K represents the set of profiles of strategies x for which the generalized game with individual constraints is defined for all agents. If x does not belong to K, it can not be a Nash equilibrium of the generalized game. Throughout the paper, for the sake of interest, we assume that K is not empty.

For a generalized game in individual constraints $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$, the best response function of agent i is defined as

$$BR_i^{Ind} : x_{-i} \mapsto \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \underset{x_i \in X_i(x_{-i})}{\arg\min} \ \theta_i(x_i, x_{-i})\} \tag{3}$$

A generalized Nash equilibrium problem (GNEP) is the given of N constrained optimization problems, that is, for each $i \in J$, given x_{-i}, agent i optimizes $\theta_i(x_i, x_{-i})$ subject to $x_i \in X_i(x_{-i})$. Note importantly that as opposed to a classical game, the best response for a generalized game depends upon the point-to-set map $X_i(x_{-i})$. A Nash equilibrium $x^* = (x_1^*, ..., x_N^*) \in K$ of the generalized game thus is such that no agent i wants to unilaterally deviate from her part of the equilibrium profile x^* but also such that the constraint of each agent $i \in J$ is satisfied, i.e., $x^* \in K$. The following definition makes clear this constraint.

Definition 3 *The profile of strategies* $x^* \in E$ *is a Nash equilibrium of the generalized game with individual constraints* $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ *if, for each* $i \in J$, $x_i^* \in BR_i^{Ind}(x_{-i}^*)$.

From the above discussion, a necessary but not sufficient condition on the profile of strategies x to be a Nash equilibrium is that $x \in K$. Since K is assumed to be not empty, it makes economic sense to require from each agent that given what the other agents are choosing, i.e., x_{-i}, agent i should pick a strategy x_i such that the profile $x = (x_i, x_{-i})$ lies in Kif such a x_i exists. Given x_{-i}, let $K_i(x_{-i})$ be the set of strategies of agent i defined as follows

$$K_i(x_{-i}) = \{x_i \in E_i : x \in K\} \tag{4}$$

We are now in a position to define a generalized game with endogenous shared constraint, that is, a generalized game in which the shared constraint is generated from the individual constraints.

Definition 4 *The 4-uplet* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *is called a generalized game with shared constraint generated from the game with individual constraints* $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. *We call such a game a game with endogenous shared constraint.*

For a generalized game in (endogenous) shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$, the best response function of agent i is defined as

$$BR_i^{Sh} : x_{-i} \mapsto \{x_i^* \text{ such that } \theta_i(x_i^*, x_{-i}) = \underset{x_i \in K_i(x_{-i})}{\arg\min}\ \theta_i(x_i, x_{-i})\} \tag{5}$$

The best response is similar to the one given for a generalized with individual constraints except that each agent i is required to choose a strategy in $K_i(x_{-i})$ rather than in $X_i(x_{-i})$. We are now in a position to give the definition of a Nash equilibrium in a game with endogenous shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$.

Definition 5 *The profile of strategies* $x^* \in E$ *is a Nash equilibrium for the generalized game with endogenous shared constraint* $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ *if for each* $i \in J$, $x_i^* \in BR_i^{Sh}(x_{-i}^*)$.

To better understand the difference between a game with individual constraints and a game with endogenous shared constraint, assume that $X_i(x_{-i}) := \{x_i \in E_i : g_i(x) \leq 0\}$ is the strategy set (or constraint) of each agent i for some function (constraint) g_i. In the literature, it is standard to describe the constraint using such functions g_i. Making use of the functions g_i, $i = 1, 2, .., N$, the admissible set K (see equation (2)) can be rewritten as $K = \{x \in E,\ \forall i \in J, g_i(x) \leq 0\}$ so that $K_i(x_{-i}) := \{x_i \in E_i : \forall j \in J, g_j(x) \leq 0\}$.

- In the game with individual constraint, given x_{-i}, each agent optimizes its objective function with respect to x_i subject to its *own individual constraint* $g_i(x) \leq 0$, that is, $x_i \in X_i(x_{-i})$.
- In the game with shared constraint generated from the individual ones, given x_{-i}, each agent optimizes its objective function with respect to x_i subject to $g_1(x) \leq 0$, $g_2(x) \leq 0$,..., $g_N(x) \leq 0$, that is, $x_i \in K_i(x_{-i})$. Given x_{-i}, when each agent i chooses a strategy x_i, she takes not only into account its own constraint but also the *constraints of all the other agents* so that (x_i, x_{-i}) lies in K.

It should be clear that in a game with endogenous shared constraint, the set of strategies of each agent i may be *reduced* compared to the game with individual constraints, that is,

$$K_i(x_{-i}) \subseteq X_i(x_{-i}) \qquad \forall i \in J \tag{6}$$

In general, the inclusion may be strict for some agents, that is, as long as $X_i(x_{-i})$ is not empty, $K_i(x_{-i}) \subset X_i(x_{-i})$. In what follows, to emphasize that the set K is the shared constraint, we may denote the game $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ as $(J, E, (\theta_i)_{i\in J}, K)$.

Let $\mathcal{N}_{Indiv}$ and $\mathcal{N}_{Shared}$ be respectively the set of Nash equilibria in individual constraints and with (endogenous) shared constraint and note that it may be the case that both sets are empty. We shall show later on that $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$ and we shall provide various illustrations of this result (2-2 games, continuous games, linear or not...).

Remark 1 *Within our approach, we take the set $X_i(x_{-i})$ (the individual constraints) as the basic premises and we derive the set K (the endogenous shared constraint) from these individual constraints. This is in sharp contrast with the literature on generalized games in which the shared constraint is exogenous. Following [Rosen, 1965], authors typically start with a classical game for which $E = \prod_{i=1}^{N} E_i$ and what they call the shared constraint is an exogenous (mathematically convenient) prescribed set $X \subset E$, that is, the profile of strategies x must be located in X, see e.g., [Fischer et al., 2014] for a nice review paper.*

Let $(J, E, (\theta_i)_{i\in J}, X)$ be a generalized game with an exogenous shared constraint. The following well-known result about the existence of Nash equilibria for concave n-person games is due to [Rosen, 1965]. Using our terminology, the following result is an existence result for a game with an exogenous shared constraint X.

Theorem 1 *([Rosen, 1965]) Let $(J, E, (\theta_i)_{i\in J}, X)$ be a game with an exogenous shared constraint where θ_i is a payoff function. If the set X is convex, closed and bounded and if each player's payoff function $\theta_i(x_i, x_{-i})$, $i \in J$ is continuous and concave in x_i, then, the generalized game has at least one Nash equilibrium.*

[Rosen, 1965] considers the simple example of a 2-person game in which X is a compact convex subset of the unit square such as an ellipse. This thus means that given the choice of agent 1, agent 2 has to choose a number between zero and one such that the couple of numbers chosen must be located in the ellipse. In many other papers, the authors consider the set X defined as $X = \{x \in E \mid G(x) \leq 0\}$, where $G : \mathbb{R}^n \to \mathbb{R}$ is a component-wise convex function called the shared constraint function, an assumption convenient for the mathematical analysis of the generalized game (see [Facchinei and Kanzow, 2010] or [Fischer et al., 2014] for review papers). An attractive feature of these games with shared constraint is that they are analytically convenient. Under particular assumptions on X and on the objective functions $\theta_i,\ i \in J$, it is often possible to use a fixed-point theorem (Kakutani, Tarski...) to prove the existence of a Nash equilibrium.

Shared constraint as binding agreements. As discussed in the introduction of this paper, sharing the constraints either requires a particular form of cooperation between agents that are called *direct agreements* in [Moulin, 1995] or an exogenous constraint that one may call a constraining regulation. In his seminal paper, [Harker, 1991] makes a similar observation. He notes that there are "two basic ways by which the constraints across players arise: the imposition of these constraints by an external 'player' in the game, or by the joint imposition by the set of N players on their own actions". Within our generalized game framework with shared constraint, given what the others do, each agent must restrict her choice in the shared constraint K, that is, given x_{-i}, each agent i

must pick a strategy x_i in $K_i(x_{-i})$ rather than in $X_i(x_{-i})$, where $K_i(x_{-i})$ is included in $X_i(x_{-i})$. Given x_{-i}, it might be the case that the best response of a given agent i lies in $X_i(x_{-i})$ and not in $K_i(x_{-i})$. As a result, this restriction to the set of strategies $K_i(x_{-i})$ rather than $X_i(x_{-i})$ requires some binding agreements (or a regulation) that are however not explicitly modelled here. This point is identical to the discussion offered in the well-known book of [Ostrom, 1990] in which she notes that one (frequent theoretical) solution to this problem is *coercion*. Roughly speaking, one can make a (perhaps disputable) distinction between two types of coercion, external or internal.

- external (or exogenous) coercion corresponds to the situation in which each agent must comply with law or regulation and this means that there is an external enforcer, to use the terminology of ([Ostrom, 1990]). In such a situation, an agent who does not comply with the law or regulation can be sanctioned (i.e., fined) by the external enforcer.
- Internal (or endogenous) coercion corresponds to the situation in which a group of agents (employees, firms, countries or even the overall society itself) must reach an agreement without any external enforcer. This clearly means that such an agreement is based on voluntarism since an agent who breach the (contractual) agreement can not be fined.

Internal coercion thus is an agreement based on self-restriction, which turns out to be very similar to the notion of *mutual restraint* discussed in [Barrett, 2007] (see chapter five). In [Barrett, 2007], the author explicitly considers the case in which agents are countries that seek to supply global public goods such as nuclear non-proliferation or climate change mitigation (e.g., limit carbon emission). For instance, when one considers a set of countries that try to reduce their individual pollution, mutual restraint can be reached through international treaties and these treaties can be thought of as an example of an internal coercion. Given x_{-i}, the restriction of each agent i to the set $K_i(x_{-i})$ can be seen as a possible formalization of the notion of mutual restraint discussed in [Barrett, 2007].

In what follows, we thus make the implicit assumption that, in shared constraint, agents succeeded to reach binding agreements so each agent (on a voluntary basis given x_{-i}) agrees to be restricted to $K_i(x_{-i})$. As we shall see, when no equilibrium exists in the game with individual constraints, this mutual restraint might be the unique solution to reach an equilibrium situation. In the last section of this paper, we shall offer two different models of collective actions in which the equilibrium only exists (depending upon parameters) in shared constraint.

3 Equilibrium analysis of 2×2 generalized games

Games with two agents and two strategies called 2×2 games are used in many introductory textbooks as a way to explain the basic concepts of game theory, but also to illustrate how in specific strategic situations, individual rationality may differ from the collective rationality (prisoners' dilemma). In [DeCanio and Fremstad, 2013], the authors apply 2×2 games to international climate negotiation problems where each agent has two strategies, Abate or Pollute. Various games such as the prisoners' dilemma and the chicken games are discussed. They also discuss cases in which there is no Nash

Cost for each agent	$\theta_A(x_A, x_B)$	$\theta_B(x_A, x_B)$
Constraint for each agent	$g_A(x_A, x_B)$	$g_B(x_A, x_B)$

Table 1: Cost and constraint for a given pair of strategies (x_A, x_B)

equilibrium in pure strategy. In [Fishburn and Kilgour, 1990] (see also [Kilgour and Fraser, 1988] and [Walliser, 1988]), they offer a more theoretical analysis to 2×2 games.

A gallery of 2×2 generalized games. Consider the following game with two agents such that each agent $i \in \{A, B\}$:

- seeks to minimize a cost function θ_i.
- can choose between two strategies 1 and 2: $E_i = \{1, 2\}$.
- is subject to the constraint $g_i(x_A, x_B) \leq c_i$

Generalized 2×2 games are more complex than classical 2×2 games encountered in economic theory (e.g., [DeCanio and Fremstad, 2013]) in which interaction occurs only through the objective function. In generalized games, whether or not the set strategies is finite, the choice of a given strategy by one agent directly impacts the objective function of the other agents but also their set of strategies (through their constraints). When one considers simple 2×2 generalized games, both the objective function and the constraint thus must be exhibited, as in table 1. In what follows, without loss of generality, we shall assume that $c_A = c_B = c$. For instance, in Fig.2, when agent A chooses strategy 1 and agent B chooses strategy 2, for agent A, the cost is equal to $\theta_A(1, 2) = 0$ and the constraint is equal to $g_A(1, 2) = 0.5$ while for agent B, the cost is equal to $\theta_B(1, 2) = 1$ and the constraint is equal to $g_B(1, 2) = 1.5$. We consider a simple taxonomy of generalized 2×2 games from the point of view of the set of Nash equilibria in individual and in shared constraint. To make things easier, we consider generalized games with a simple structure: as long as agents are "coordinated", when they choose the same strategy, their cost is equal to one, that is, $\theta_i(1, 1) = \theta_i(0, 0) = 1$ for $i \in \{A, B\}$ and their constraint is satisfied. We consider three different games in which we only change the cost or the constraint when agents are not coordinated. Recall that $\mathcal{N}_{Indiv}$ and $\mathcal{N}_{Shared}$ are respectively the set of Nash equilibria in individual constraints and with (endogenous) shared constraint

1. $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared}$. Consider Fig. 1 and assume that $c = 1$. There are two Nash equilibria in individual constraints, $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared} = \{(1, 1); (2, 2)\}$. Knowing that agent A chooses strategy 1, the best response of agent B is to choose strategy 1. Conversely, knowing that agent B chooses strategy 1, the best response of agent A is to choose strategy 1 so that the pair of strategy $(1, 1)$ is a Nash equilibrium in individual constraints. The constraint of each agent is satisfied so that the pair of strategy $(1, 1)$ is also a Nash equilibrium in shared constraint. A similar analysis can be done for the pair of strategies $(2, 2)$. For such a game, everything is

		B			
		1		2	
A	1	1	1	2	1
		0.5	0.5	0.5	1.5
	2	1	2	1	1
		1.5	0.5	0.5	0.5

Figure 1:

		B			
		1		2	
A	1	1	1	0	1
		0.5	0.5	0.5	1.5
	2	1	0	1	1
		1.5	0.5	0.5	0.5

Figure 2:

		B			
		1		2	
A	1	1	1	2	0.75
		0.5	1.5	1.5	0.5
	2	1.5	0.5	1	1
		1.5	0.5	0.5	0.5

Figure 3:

Agent B \ Agent A	1	2
1	$(1, Y)$	(a, D_A, D_B)
2	(b, E_A, E_B)	$(1, Y)$

Table 2: Symmetric payoff game 2×2 generalized game with asymmetric constraint

as if the constraint were not an issue since it is always satisfied. Such a game can be thought of as a coordination (generalized) game since there are two identical Pareto optimal Nash equilibria.

2. $\mathcal{N}_{Indiv} \subset \mathcal{N}_{Shared}$. Consider Fig. 2 and assume that the constraint is $c = 1$. In such a case, only $(1, 1)$ is a Nash equilibrium in individual constraints so that $\mathcal{N}_{Indiv} = \{(1, 1)\}$. To see this, assume that agent A chooses $x_A = 1$. Abstracting the constraint, agent B is indifferent between strategy 1 and 2. However, since $g_B(1, 1) = 0.5$ and $g_B(1, 2) = 1.5$, the constraint of agent B is only satisfied when she/he chooses strategy 1, i.e., $X_B(x_A = 1) = \{1\}$. It thus follows that the best response of agent B is $BR_B^{Ind}(x_A = 1) = 1$. Assume now that agent B chooses strategy 1. Since agent A only satisfies the constraint for decision 1, it thus follows that $BR_A^{Ind}(x_B = 1) = 1$ so that $(1, 1)$ is a Nash equilibrium under individual constraints. Consider now the pair of strategies $(2, 2)$. If agent B chooses $x_B = 2$, then, the best response in individual constraint is $BR_A^{Ind}(x_B = 2) = 1$ and this means that $(2, 2)$ is not a Nash equilibrium under individual constraints. However, the pair of strategies $(2, 2)$ is a Nash equilibrium in shared constraint since the best response in shared constraint are $BR_A^{Sh}(x_B = 2) = 2$ and $BR_B^{Sh}(x_A = 2) = 2$. To see this, it suffices to note that $X_A(x_B = 2) = \{1, 2\}$ while $K_A(x_B = 2) = \{2\}$ and $X_B(x_A = 2) = \{1, 2\}$ while $K_B(x_A = 2) = \{2\}$. It thus follows that $\mathcal{N}_{Shared} = \{(1, 1); (2, 2)\}$ while $\mathcal{N}_{Indiv} = \{(1, 1)\}$

3. $\mathcal{N}_{Indiv} = \emptyset$ while $\mathcal{N}_{Shared} \neq \emptyset$. Along the same line of reasoning, it is easy to see in Fig. 3 that there is no Nash equilibrium in individual constraint while the pair of strategies $(2, 2)$ is a Nash equilibrium in shared constraint. In the game in individual constraints, when agent A chooses strategy 2, agent B chooses strategy 1 so that $(2, 2)$ is not a Nash equilibrium. However, in shared constraint, agent B can not choose strategy 1 because agent A would not satisfy her/his constraint anymore.

On 2×2 generalized games with symmetric payoff but asymmetric constraint. Consider once again the coordination game (see table 2) in which the payoff of each agent is equal to one when they choose the same strategy, i.e., $(1, 1)$ or $(2, 2)$. Without loss of generality, we shall use a dummy variable for the constraint, Y (Yes) when it is satisfied and N (No) when it is not. We thus have four dummy variables, D_A, D_B, E_A, E_B. In principle, one may derive many simple results from such a game. We shall consider two. One in which the game reduces to a classical game, i.e., the constraints are irrelevant so that the set of Nash equilibria are identical in individual and in shared constraint, and another one in which there is no Nash equilibrium in individual constraint while there is at least one equilibrium in shared constraint.

Fact 1 . *Consider a symmetric payoff* 2×2 *generalized game with asymmetric constraint as given in table 2 and let* D_A, D_B, E_A, E_B *be four dummy variables.*

1. *If* $D_A = D_B = Y$ *and* $E_A = E_B = Y$ *and* $a > 1$, $b > 1$, *then,* $\mathcal{N}_{Indiv} = \mathcal{N}_{Shared} = \{(1,1);(2,2)\}$ *and the game reduces to a classical coordination game.*

2. *If* $D_A = E_A = Y$ *and* $E_B = D_B = N$ *(or if* $D_A = E_A = N$ *and* $E_B = D_B = Y$*) and* $a < 1$, $b < 1$, *then,* $\mathcal{N}_{Indiv} = \emptyset$ *while* $\mathcal{N}_{Shared} = \{(1,1);(2,2)\}$

We leave the proof and all the other cases to the interested reader. When the constraint is always satisfied for each agent and when $a > 1$ and $b > 1$, it is in the interest of both agents to be perfectly coordinated and the game reduces to a pure coordination game with two Nash equilibria in individual constraints and in shared constraint since the constraint plays no role. When $a < 1$, $b < 1$ and when the constraint is always satisfied for one agent but is no satisfied for the other agent when their decisions are not coordinated, then, there are still two equilibria in shared constraint but no equilibrium in individual constraint.

4 Theoretical results

4.1 An elementary general result

As we shall now see, a striking feature of a game with endogenous shared constraint, compared with the game with individual constraints, is that it may possess *additional Nash equilibria*. We already know that for a profile x to be a Nash equilibrium, x must be in K. But if each agent i, given x_{-i}, agrees to choose x_i such that $x = (x_i, x_{-i})$ is in K (i.e., $x_i \in K_i(x_{-i})$), the set of Nash equilibria may be *larger*.

Proposition 1 *The set of Nash equilibria of a game with individual constraints denoted* $\mathcal{N}_{Indiv}$ *is included in the set of the Nash equilibria of the game with shared constraint generated from the individual constraints, denoted* $\mathcal{N}_{Shared}$*; that is,* $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$.

Proof. See the appendix.

Independently, [Feinstein and Rudloff, 2021] proved a similar result (see their Theorem 3.2). While easy to prove from a mathematical point of view, it is important to point out that such a result only makes sense for generalized games in which the shared constraint is derived from the individual ones. As we shall see, this result is particularly relevant when $\mathcal{N}_{Indiv} = \emptyset$ while $\mathcal{N}_{Shared} \neq \emptyset$. To illustrate the above proposition, that is, to show that $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$, consider the following game in which the set of strategies have the cardinality of the *continuum*.

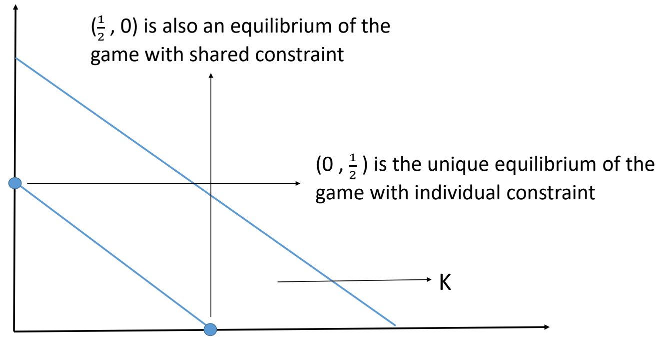

Figure 4: One equilibrium in individual constraints, an infinity of equilibria in shared constraint

Let $J = \{1, 2\}$ be the set of agents and let $E_1 = E_2 = [0, 1]$ be the strategy set of each agent so that $E = [0, 1] \times [0, 1]$ is the strategy space. The cost functions are $\theta_1(x_1, x_2) = x_1 + x_2$ and $\theta_2(x_1, x_2) = x_2 - x_1$. Assume now that the individual constraint for agent 1 is $g_1(x_1, x_2) = x_1 + x_2 \leq 1$ while it is equal to $g_2(x_1, x_2) = x_1 + x_2 \geq \frac{1}{2}$ for agent 2. In this example, the set K is defined as

$$K = \left\{ (x_1, x_2) \in [0, 1]^2 : x_1 + x_2 \leq 1 \text{ and } x_1 + x_2 \geq \frac{1}{2} \right\} \tag{7}$$

It is not difficult to see that K is a compact and convex set (see Fig. 4) and that the profile of strategies $(0, \frac{1}{2}) \in K$ is a Nash equilibrium of the game with individual constraints. Consider now the profile of strategies $(\frac{1}{2}, 0) \in K$.

- In the game with individual constraints, $(\frac{1}{2}, 0) \in K$ is *not* a Nash equilibrium. To see this, it suffices to note that when $x_2 = 0$, the best response of agent 1 is 0 and the constraint of agent 1 is fulfilled since $g_1(x_1, x_2) = x_1 + x_2 \leq 1$. Since $\frac{1}{2}$ is not the best response, $(\frac{1}{2}, 0) \in K$ thus is not an Nash equilibrium.

- In the game with shared constraint, $(\frac{1}{2}, 0) \in K$ is a Nash equilibrium. To see this, note that if $x_1 = \frac{1}{2}$, then, the best response of agent 2 is 0. This choice of 0 minimizes the cost of agent 2 and satisfies the constraint of agent 2 since $\frac{1}{2} + 0 \geq \frac{1}{2}$ but also the constraint of agent 1 since $\frac{1}{2} + 0 \leq 1$. If $x_2 = 0$, the best response of agent 1 is now to choose $\frac{1}{2}$ because agent 1 takes into account the constraint of agent 2, i.e., $g_2(x_1, x_2) = x_1 + x_2 \geq \frac{1}{2}$. As opposed to the game with individual constraints, agent 1 can not choose 0. As a result, $(\frac{1}{2}, 0)$ is a Nash equilibrium for the game with endogenous shared constraint. Actually all vectors (x_1, x_2) such that $x_1 + x_2 = \frac{1}{2}$ are Nash equilibria in shared constraint. Indeed, if $x_1 + x_2 = \frac{1}{2}$ both constraints are satisfied and no agent has an incentive and possibility to move to a better state. Let us prove that the converse is true. Consider (x_1, x_2) such that $1 \geq x_1 + x_2 > \frac{1}{2}$ then one of the two agents will have the incentive and possibility to decrease its cost. Therefore the set of Nash equilibria in endogenous shared constraint for this game is equal to: $\mathcal{N}_{Shared} = \{(x_1, x_2) \in E,\ x_1 + x_2 = \frac{1}{2}\}$.

- Let us now show that $\mathcal{N}_{Indiv} = \{(0, \frac{1}{2})\}$. From Proposition 1 we have that $\mathcal{N}_{Indiv} \subset \mathcal{N}_{Shared} = \{(x_1, x_2) \in E,\ x_1 + x_2 = \frac{1}{2}\}$. Consider $(x_1, x_2) \in \mathcal{N}_{Shared}$ such that $x_1 > 0$. Then agent 1 would have an incentive and possibility to take $x_1 = 0$ instead and therefore (x_1, x_2) would not be a Nash equilibrium in individual constraints. Therefore, $\mathcal{N}_{Indiv} = \{(0, \frac{1}{2})\}$.

This is an example of game in which there is a single equilibrium in individual constraints and an infinity of equilibria in shared constraint. As already said, by requiring from each agent i to choose a strategy in $K_i(x_{-i}) \subseteq X_i(x_{-i})$, this may *expand* the set of Nash equilibria. Proposition 1 thus yields the two following basic insights.

1. There may be situations in which there is no Nash equilibrium in the game with individual constraints, that is, $\mathcal{N}_{Indiv} = \emptyset$ while there exists a Nash equilibrium in the game with endogenous shared constraint (generated from the individual ones), that is $\mathcal{N}_{Shared} \neq \emptyset$.

2. If no Nash equilibrium exists in the game with endogenous shared constraint, that is, if $\mathcal{N}_{Shared} = \emptyset$, no Nash equilibrium exists in the game with individual constraints, that is, $\mathcal{N}_{Indiv} = \emptyset$, the converse is however not true.

While proposition 1 is elementary to prove from a mathematical point of view, it is interesting to note that it does not require the underlying functions to be differentiable, as opposed to the variational formulation of the Nash equilibrium in generalized games. It can also be applied to generalized games in which the set of strategies of each agent is finite, for which the objective function needs not be a continuous function (see the gallery of 2×2 games).

Let us now provide an additional example in which $\mathcal{N}_{Indiv} = \emptyset$ while $\mathcal{N}_{Shared} \neq \emptyset$.

Let $J = \{1, 2, ..., N\}$ be the set of agents. For each $i \in J$, the strategy set of agent i is $E_i = [0, 1]$ so that $E = [0, 1]^N$. Assume moreover that the characteristics of the agents are as follows.

1. Each agent $i \in J$ has a cost function (to be minimized) equal to $\theta_i(x_i) = x_i$.

2. Each agent $i \in J$ has the following constraint function.

   - If $x_j \geq 0.9$ for every $j \in J \setminus \{i\}$, then, $X_i(x_{-i}) = [\frac{1}{2}, 1]$.
   - If $x_j < 0.9$ for at least one $j \in J \setminus \{i\}$, then, $X_i(x_{-i}) = \emptyset$.

This game thus differs from classical ones encountered in economic theory in that the interaction only occurs through the set of strategies but not through the objective functions. From the specification of the game, if a given agent i chooses a number $x_i \geq 0.9$, the cost of agent i is simply equal to the number x_i chosen. If there is one agent j who chooses a number $x_j < 0.9$, with $i \neq j$ then, the set of strategies of a given agent is empty and the objective function thus is undefined[4]. Before

[4] One can think of this abstract game theoretic framework to represent a tax competition problem in a fiscal union (see for instance [Zodrow, 2003]). State members may be committed to choose a tax rate greater than a given threshold. However, if one member state breaches the commitment, the problem becomes undefined for the other member states in that their strategy set is empty.

discussing the outcome of the game with individual constraints, let us consider the game with shared constraint generated from the individual constraints. In this game with shared constraint, the set of strategies of each agent $i \in J$ is equal to $K_i(x_{-i}) = [0.9, 1]$ if $x_{-i} \in [0.9, 1]^{N-1}$ (if $x_{-i} \notin [0.9, 1]^{N-1}$, then, $X_i(x_{-i}) = \emptyset$ so that $K_i(x_{-i}) = \emptyset$) so that K is not empty and equal to $K = [0.9, 1]^N$. Since agents minimize a cost function, the profile of strategies $x^* = (0.9, ..., 0.9)$ thus is the *unique* Nash equilibrium of the game with shared constraint $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J})$. Therefore, from proposition 1, if the game with individual constraints $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ has a Nash equilibrium, it must necessarily be $x^* = (0.9, ..., 0.9)$. But $x^* = (0.9, ..., 0.9)$ is *not* a Nash equilibrium of the game with individual constraints. If $x_{-i} = (0.9, ..., 0.9)$, the best response of agent i is 0.5 and not 0.9, which means that $x^* = (0.9, ..., 0.9)$ is not a Nash equilibrium of the game with individual constraints. When one agents picks a number lower than 0.9, the objective functions are undefined for the other agents and this means that there is no Nash equilibrium in such a game with individual constraints.

4.2 Existence and characterization results for non-classical games

Classification. In a generalized game, (see table 3), interaction occurs not only through the objective function but also through the strategy sets, that is, the objective function of a given agent $i \in J$ depends upon the decisions of the other players, i.e., $\theta_i(x_i, x_{-i})$ but the set of strategy of each agent i also depends upon the decisions of the other agents, that is, each agent i must choose $x_i \in X_i(x_{-i})$ where $X_i(x_{-i})$ is a subset of E_i. In table 2, we offer a fruitful classification of games, encountered in Economics and Operations Research literature, through the way interaction is introduced.

In table 3, for what we call *classical games*, those encountered in most economic textbooks and papers in economic theory (see e.g. classic textbooks [Fudenberg and Tirole, 1991], [Moulin, 1986], [Myerson, 2013], [Osborne and Rubinstein, 1994]), interaction occurs through the objective functions but not through the constraints. For what we call *non-classical games*, interaction occurs through the constraints but not through the objective functions. While these non-classical games are fairly natural when one thinks to collective action problems, their application in the Economic literature is still limited at the moment. In the future, these kind of generalized games could be widely applied. Finally, when there is no interaction at all, that is, when the objective function only depends upon the decision variable of agent i, that is $\theta_i(x_i)$ and when E_i is exogenously given (e.g., it is a compact set), this gives rise to a non-strategic decision problem.

Existence and characterization results for non-classical games. We now provide an existence result which can be thought as a generalization of Rosen's theorem for non-classical game. To the best of our knowledge, such a non-classical game appears for the first time in [Braouezec and Wagalath, 2019] and more recently in [Braouezec and Kiani, 2022] in a financial stress test framework. Another framework is provided in [Banerjee and Feinstein, 2021] although the authors do not explicitly formulate their problem as a generalized game.

Dependence through the objective function and/or through the strategy sets

Objective function \ Strategy set	Yes	No
Yes	Generalized game	Classical game
No	Non-Classical game	Non-strategic decision problem

Table 3: Types of interaction in games

Proposition 2 *Consider a non-classical game $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ such that for each $i \in J$, $\theta_i(x) = \theta_i(x_i)$ and assume that for each $i \in J$, θ_i is continuous and that K is compact. The following results hold.*

1. *The game with shared constraint $(J, E, (\theta_i)_{i\in J}, K)$ admits at least one Nash equilibrium.*
2. *The set of Nash equilibria in shared constraint $\mathcal{N}_{Shared}$ exactly coincides with the set of minimizers of the total cost function $\sum_{i\in J} \theta_i(x_i)$ on K and each Nash equilibrium is Pareto optimal.*

Proof. See the appendix.

Note that from the above result, one cannot infer any result regarding the existence of Nash equilibria in individual constraints. As the next proposition shows, under these conditions, a Nash equilibrium in individual constraints does not always exist. Note interestingly that in proposition 2, we make no assumption regarding the evolution of θ_i with respect to $x_i \in \mathbb{R}^d$ with $d \geq 1$. We shall now show an example in which the non-classical game satisfies the assumptions of Proposition 2, and therefore the game with endogenous shared constraint $(J, E, (\theta_i)_{i\in J}, K)$ has at least one Nash equilibrium, but the initial game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ may or may not have a Nash equilibrium. The following result holds for non-classical games with positive linear constraints that will be illustrated in Section 5.1. Note that in the next result, the objective function is typically a profit function to be maximized.

Proposition 3 *Consider a non-classical game $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ such that for each $i \in J$, $E_i = \mathbb{R}_+$ and let $S_i \in \mathbb{R}^+$ be the constraint of agent i. Assume that the payoff function $\theta_i(x) = \theta_i(x_i)$ is an increasing, continuous and concave function of x_i and that the constraint of agent i is given by a linear function $l_i(x) = \sum_{j\in J} a_{ij}x_j$ where $a_{ij} > 0$ for each i and j, such that $X_i(x_{-i}) = \{x_i \in E_i,\ l_i(x) \leq S_i\}$. Let $A = (a_{i,j})_{i,j\in J}$ be a matrix and $S = (S_i)_{i\in J}$ be a column vector.*

1) Game in individual constraints:

- *If the linear system $Ax = S$ admits no solution in $\mathbb{R}^N_+$, then there is no Nash equilibrium in individual constraints.*
- *If the matrix A is invertible, then, there exists a unique solution x^* to the linear system and if x^* is in $\mathbb{R}^N_+$, there exists a unique Nash equilibrium of the form $x^* = A^{-1}S$. If x^* is not in $\mathbb{R}^N_+$, there is no Nash equilibrium.*

- *If the matrix A is not invertible and the linear system $Ax = S$ admits at least one solution $x^{*,0} \in \mathbb{R}_+^N$, then the linear system admits infinitely many solutions, and there are infinitely many Nash equilibria in individual constraints given by the set* $\{x^{*,0} + y, y \in \ker(A)\} \cap \mathbb{R}_+^N$.

2) Game in shared constraint:

- $\mathcal{N}_{Indiv} \subseteq \mathcal{N}_{Shared}$ *(this is a direct consequence of Proposition 1)*
- *There always exists at least one Nash equilibrium.*

Proof. See the Appendix.

Remark 2 *This result is obviously true for the symmetric case of a game with agents seeking to minimize a cost function $Cost_i(x) = Cost_i(x_i)$ decreasing continuous and convex function of x_i and with individual constraints $X_i(x_{-i}) = \{x_i \in E_i,\ l_i(x) \geq S_i\}$ with $l_i(x) = \sum_{j \in J} a_{ij} x_j$.*

We provide in appendix two proofs of the second point of the above proposition regarding the existence of a Nash equilibrium in shared constraint (part 2). We rely on Rosen theorem for one proof but we use proposition 3 for the second proof. It suffices to note that the game is non-classical, the shared constraint K is a compact set and the cost functions are continuous. Therefore, this game satisfies the assumptions of Proposition 2, and a Nash equilibrium in shared constraint always exist.

5 Applications to collective action problems

We shall now discuss two applications of games with endogenous shared constraint, one applied to an environmental problem and another applied to a public good problem. A striking feature of these two economic applications is that a Nash equilibrium may not exist in individual constraints while it always exists in endogenous shared constraint. The first example is applied to an environmental problem (it is indeed formulated as a non-classical game) and the existence result can be proved either with the fundamental theorem of [Rosen, 1965] (see theorem 1 in this paper) or with Proposition 2. In the second example, applied to the financing problem of a public good, we consider a generalized game and show that when the dispersion of the optimal contribution of each agent is too large, no Nash equilibrium exists in individual constraints while a Nash equilibrium exists in endogenous shared constraint. From a mathematical point of view, to ease the analysis, we shall focus on the case in which the individual constraints are uni-dimensional.

5.1 Limiting global warming

Given the critical nature of the subject, the limitation of global warming of the earth, there is a large (game theoretical based) literature on the subject (see e.g., [Hoel and Schneider, 1997], [Carraro and Siniscalco, 1993], [Barrett, 2001]). We refer the reader to [Missfeldt, 1999], which is a survey of game theoretic models of trans-boundary pollution but note that this review paper does not mention generalized games. We found only few papers on the subject, [Tidball and Zaccour, 2005] and [Krawczyk, 2005], that analyze the pollution problem as a generalized game. In these models, the choice variable of a given country x_i is typically its pollution level (i.e., measured by the volume of emission of greenhouse gas) which, in the simplest case, is modeled as a linear function of its production. Such an environmental problem is interesting but challenging because the production of a given country typically generates *negative externalities* (i.e., pollution) to all the other countries.

In [Tidball and Zaccour, 2005], they consider a model where each country seeks to maximize a profit function of the type $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + ... + x_n)$, where x_i represent the emissions of country i and are assumed to be proportional to the production of country i, $f_i(x_i)$ is a non-negative, twice-differentiable, concave and increasing function, and the damage cost due to all the countries is denoted by a convex twice-differentiable increasing cost function $d_i(x_1 + ... + x_n)$. They consider three types of problems which correspond to three types of constraint:

- A *Generalized Nash equilibrium problem with individual constraints* where each agent is seeking to maximize $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + ... + x_n)$ subject to the constraint $x_i \leq T_i$ with T_i an exogenous given upper bound threshold on emissions.

- A *cooperative scenario*, where agents agree to jointly maximize the sum of their profit function: $\max_{x_1,...,x_n} \sum_{i=1}^n (f_i(x_i) - d_i(x_1 + x_2 + ... + x_n))$ subject to $\sum_{i=1}^n x_i \leq \sum_{i=1}^n T_i$.

- A *Generalized Nash equilibrium problem with an exogenous shared constraint* where each agent is seeking to maximize $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + x_2 + ... + x_n)$ subject to the constraint $\sum_{i=1}^n x_i \leq \sum_{i=1}^n T_i$.

The purpose of [Tidball and Zaccour, 2005] is to characterize and compare the solutions of these three different scenarios. They show that a Nash equilibrium in individual constraints may be better than a Nash equilibrium with (exogenous) shared constraint. In their framework, the shared constraint is actually not generated from the individual constraints[5]

In the same vein, [Krawczyk, 2005] proposes another model where three players $j = 1, 2, 3$ located along a river are engaged in an economic activity at a chosen level x_j and their joint production externalities must satisfy environmental constraints set by a local authority. It is assumed that player$j$ has a level of pollution $e_j x_j$, where e_j is the emission coefficient of player j. The pollution is expelled into the river and reaches a monitoring station in the amount of $\sum_{j=1}^3 \delta_{jl} e_j x_j$ where δ_{jl} is the decay-and-transportation coefficient from player j to location l, and it is assumed that there are

[5] The individual constraint of country i does not depend upon the choices of the other countries, for each $i \in J$, $X_i(x_{-i}) = [0, T_i]$ no matter what x_{-i} is. It thus follows that $K = \prod_{i \in J} X_i$ while the shared constraint in [Tidball and Zaccour, 2005] is given by the set $S := \{x \in \mathbb{R}_+^n : \sum_{i \in J} x_i \leq \sum_{i \in J} T_i\}$. This shared constraint thus is *not* generated from the game with individual constraints since x can be in S but not in K. To see this within a numerical example, assume that $n = 2$ and that $T_1 = 3$ and $T_2 = 2$ so that $X_1 = [0, 3]$ while $X_2 = [0, 2]$. The point $(2, 3) \notin X_1 \times X_2$ while $(2, 3) \in S$.

two monitoring stations, $l = 1, 2$, and the local authority has set maximum pollutant concentration levels C_l. It gives the following Generalized Nash equilibrium problem with shared constraint: each player j is supposed to maximize its net profit $\phi_j(x) = d_1 - d_2(x_1 + x_2 + x_3) - (c_{1j} + c_{2j}x_j)x_j$ (where $d_1, d_2, c_{1j}c_{2j}$ are given constants) under the shared constraint $\sum_{j=1}^{3} \delta_{jl} e_j x_j \leq C_l$, $l = 1, 2$. [Krawczyk, 2005] proves existence of a unique Nash equilibrium under reasonable assumptions and exhibits an algorithm that converges towards the solution.

Inspired by [Tidball and Zaccour, 2005] and [Krawczyk, 2005], we now introduce a new environmental model formulated in terms of generalized game with individual constraints and with endegenous shared constraint.

Consider the following model with $N \geq 2$ countries/economies. Let $x_i \in \mathbb{R}_+$ be the quantity of non-renewable energy (i.e., fossil energy) chosen by country i where the carbon emissions C_i by country $i \in \{1, 2, ..., N\}$ are proportional to x_i (similar to [Tidball and Zaccour, 2005] and [Krawczyk, 2005]), that is $C_i = c_i \times x_i$, where $c_i > 0$ (and note that $E = \mathbb{R}_+^N$). Let $f_i(x_i)$ be the profit function of country i as a function of x_i where f_i is an increasing continuous and concave function. On this aspect, we differ from [Tidball and Zaccour, 2005] and [Krawczyk, 2005] since we assume that the payoff function of a country only depends on x_i, the quantity of non-renewable energy chosen by i. We assume that each country i has to satisfy an individual linear constraint of the form:

$$g_i(x) := a_{ii} \times c_i \times x_i + \sum_{j \neq i} a_{ji} \times c_j \times x_j \leq S_i \tag{8}$$

where a_{ji} is a coefficient measuring how the use of fossil fuels by country j is impacting the environment of country i and S_i is a threshold determined for each country i. The constraint S_i may come from a regulatory institution. Note that in the case in which $j = i$, the coefficient a_{ii} measures how the use of fossil fuels by country i is impacting its own environment. Given x_{-i}, the aim of country i is to maximize $f_i(x_i)$ subject to $g_i(x_i, x_{-i}) \leq S_i$ and note that, using our classification of games, such a strategic interaction is an example of a non-classical game.

The aim is now to study the possible existence of a Nash equilibrium for this game with individual constraints and its generated game with (endogenous) shared constraint. As seen earlier, it may indeed be the case that there is no Nash equilibrium for the game with individual constraints. To see this, assume that $N = 2$. Consider country 1 and assume that $a_{11} = c_1 = 1$ and that $a_{21} = 0$ so that its constraint is given by $x_1 \leq S_1$. Consider now country 2 and assume that $a_{22} = c_2 = 1$ and assume now that $a_{12} = 1$. From equation (8), it thus follows that the constraint of country 2 is given by $x_2 + x_1 \leq S_2$. Assume now that $S_1 > S_2$ and that this generalized game with individual constraints has at least a Nash equilibrium (x_1^*, x_2^*). Such an equilibrium must satisfy $x_1^* = S_1$. But in such a case, no matter what $x_2 \geq 0$ is, the constraint of country 2 is never satisfied which means that there is no Nash equilibrium in individual constraints.

We shall now derive a more general result about the (non) existence of a Nash equilibrium in this environmental game theoretical model with individual constraints. From the individual constraint given by equation (8), for a profile of strategy $x \in \mathbb{R}_+^N$ to be Nash equilibrium, the constraints must be satisfied, that is, $x \in \mathbb{R}_+^N$ must be such that $g_i(x) \leq S_i$ for $i = 1, 2, ..., N$. From the monotonicity of the profit function of each country i, the constraint of each country will be binding at equilibrium $x^* = (x_1^*, ..., x_N^*)$, which means that for $i = 1, 2, ..., N$, $g_i(x^*) = S_i$. The problem reduces to the analysis of a linear system of the form $A^\top \text{diag}(c)X = S$ where A is the matrix of the coefficient $(a_{ij})_{i,j=1,2,...,N}$ ($\top$ indicates the transpose), $\text{diag}(c) = \text{diag}(c_1, ..., c_N)$, $x = (x_1, .., x_N) \in \mathbb{R}_+^N$ and S is the vector formed by S_i, $i = 1, 2, ..., N$. If $x^* = (x_1^*, ..., x_N^*)$ is a Nash equilibrium for this game with individual constraints, it must satisfy the linear system $A^\top \text{diag}(c)x^* = S$, that is:

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1N} \\ a_{22} & a_{22} & ... & a_{2N} \\ ... & ... & ... & ... \\ a_{N1} & a_{N2} & ... & a_{NN} \end{pmatrix}^T \begin{pmatrix} c_1 & 0 & ... & 0 \\ 0 & c_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & c_N \end{pmatrix} \begin{pmatrix} x_1^* \\ x_2^* \\ ... \\ x_N^* \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix} \tag{9}$$

where $x^* \in \mathbb{R}_+^N$.

The next result exhibits conditions under which there is no Nash equilibrium in individual constraints. In particular, when the above linear system has no solution in $\mathbb{R}_+^N$, then, no Nash equilibrium of the game with individual constraints can exist. However, in such a case in which the linear system has no solution, a Nash equilibrium of the game with endogenous shared constraint still exists.

Proposition 4 *The game with individual constraints has (at least) one Nash equilibrium x^* if and only if the linear system as defined in equation (9) admits at least one solution $x^* \in \mathbb{R}_+^N$. If the linear system given by equation (9) admits no solution in $\mathbb{R}_+^N$, then, the game with individual constraints has no Nash equilibrium.*

Proof. See the appendix.

In appendix, we give a complete characterization of the existence or not of Nash equilibria for the game with individual constraints. It is interesting at this stage to compare our framework with the one offered in [Tidball and Zaccour, 2005] and [Krawczyk, 2005]. Within our approach, as opposed to the two mentioned papers, a Nash equilibrium does not always exist in individual constraints. However, as we shall see, a Nash equilibrium in shared constraint always exists. In [Tidball and Zaccour, 2005], they show that Nash equilibria in individual constraints may be better in some sense than the ones in shared constraint but, as already discussed, their shared constraint is exogenous since it is not generated from the individual ones.

Let us now consider the game with shared constraint generated from our game with individual constraints in which each country i is seeking to maximize $f_i(x_i)$ subject to the endogenous shared constraint defined as $g_k(x) := a_{kk} \times c_k \times x_k + \sum_{j \neq k} a_{jk} \times c_j \times x_j \leq S_k$, $k = 1, ..., N$. From proposition 1, we know that the Nash equilibria of the game with shared constraint contains the Nash equilibria of the game with individual constraints. But Nash equlibria of the game with individual constraints may not always exist, as seen in proposition 4. In the next result, we state that there always exists Nash equilibria for our game with endogenous shared constraint. This is a direct consequence of proposition 3.

Proposition 5 *The game with a shared constraint generated from the individual ones always has at least one Nash equilibrium. In particular, a Nash equilibrium in the game with endogenous shared constraint exists whether the linear system given by equation (9) admits a solution or not.*

Proposition 5 shows that a Nash equilibrium always exists in the game with shared constraint generated from the individual constraints while it may fail to exist in the game with individual constraints. By reinforcing the environmental constraints, i.e., by sharing the constraints, each country takes into account the individual constraints of the other countries, and this binding agreement always generates at least one Nash equilibrium.

5.2 Contributing to a public good

We now consider a model of collective action with $N \geq 2$ agents, similar to [Guttman, 1978] and to [Cornes and Hartley, 2007] (see also [Buchholz et al., 2011]) but in which each agent faces an individual constraint (see [Sandler, 2015] for a review paper on collective action models). Each agent i is asked to make a voluntary contribution to finance a public investment project such as a bridge, a street lighting or any public infrastructure equipment (water, gas, internet...) subject to an individual constraint. For concreteness, we assume that the higher the total contributions, the higher quality of the underlying investment. Following the standard terminology introduced in [Guttman, 1978] (see also ([Buchholz et al., 2011])), let $x_i \in E_i := \mathbb{R}_+$ be the flat (or direct) contribution of agent i and let $b \in [0,1]$ the known percentage of the sum of the flat contributions of all the other agents. The indirect contribution of a given agent i is by definition equal to $b\sum_{j\neq i} x_j$, so that the total contribution of agent i is equal to $x_i + b\sum_{j\neq i} x_j$. Following [Guttman, 1978], the utility of each agent i is assumed to be equal to

$$U_i(x_i, x_{-i}) = v_i(x_i + b\sum_{j\neq i} x_j) - (x_i + b\sum_{j\neq i} x_j) \tag{10}$$

where the function $v_i(.)$ measures the willingness-to-pay of an agent i for the public project and depends upon her own contribution but also upon the contributions of all the other agents. As in the literature on collective actions and aggregative games ([Guttman, 1978], [Cornes and Hartley, 2007] [Chen and Zeckhauser, 2018], [Cornes and Hartley, 2012], see also [Cornes, 2016] for a recent review paper), the willingness-to-pay $v_i(.)$ depends upon x_{-i} only through the sum of the contributions of the other agents. Given a profile of strategies $x = (x_i, x_{-i})$, let

$$z_i = x_i + b\sum_{j\neq i} x_j \tag{11}$$

be the total contribution of agent i and note that the utility function of agent i can be written as a function of the sole scalar z_i.

$$U_i(x_i, x_{-i}) = v_i(z_i) - z_i \tag{12}$$

We shall assume that $v_i(z_i)$ is a twice continuously differentiable increasing and strictly concave function of z_i (with $v_i(0) = 0$) so that U_i is also a strictly concave function[6]. An example of a functional form for $v_i(z_i)$ is $v_i(z_i) = \xi_i \sqrt{z_i}$ where ξ_i is a positive scalar.

Regarding now the individual constraints, it can be formulated as a *budget constraint* or as a *reservation utility*.

- The budget constraint of agent i can be given as an exogenous revenue of agent i, r_i, which means that it must be the case that $z_i \leq r_i$.
- The reservation utility constraint can be given as an exogenous reservation utility of agent i, $\overline{u}_i$, which means that it must be the case that $U_i(x_i, x_{-i}) \geq \overline{u}_i$. The threshold $\overline{u}_i$ can be interpreted as a minimal quality for the public good according to agent i.

The optimization problem of a given agent i can be formulated as a utility maximization problem subject to a budget constraint and/or subject to a reservation utility constraint. Let

- $\mathcal{B} \neq \emptyset$ be the subset of J that are subject to a budget constraint.
- $\mathcal{U} \neq \emptyset$ be the subset of J that are subject to a utility constraint.

where the $\mathcal{B}$ and $\mathcal{U}$ are assumed to form a partition of J, that is, $\mathcal{B} \cup \mathcal{U} = J$, with $\mathcal{B} \cap \mathcal{U} = \emptyset$. Since each agent i is endowed with a utility function assumed to be a concave function of z_i, it makes thus sense to consider her *ideal contribution* z_i^* to the public project.

Let $z_i^* = \arg\max_{z_i \geq 0} U_i(z_i) := v_i(z_i) - z_i$ subject to a budget constraint if $i \in \mathcal{B}$ and a reservation utility constraint if $i \in \mathcal{U}$. Given the assumptions on the utility function, z_i^* is unique and for the sake if interest, we assume that $z_i^* > 0$ for each $i \in J$. .

Let $\overline{z}_i$ be the critical threshold of each agent $i \in J$. If $i \in \mathcal{B}$, then $\overline{z}_i = r_i$. Consider now agent $i \in \mathcal{U}$. Given the assumptions on the utility function U_i (continuity, strict concavity), there exists two critical thresholds $\underline{z}_i < z_i^*$ and $\overline{z}_i > z_i^*$ such that $U_i(\underline{z}_i) = v_i(\underline{z}_i) - \underline{z}_i = \overline{u}_i = U_i(\overline{z}_i) = v_i(\overline{z}_i) - \overline{z}_i$. As before, let us denote BR_i^{Ind} the best response function in individual constraints. The following fact holds and is an elementary consequence of the definition of z_i^*.

Fact 2 *The best response of agent $i \in J$ is given below.*

- *If $b \sum_{j \neq i} x_j \leq z_i^*$, then, $BR_i^{Ind}(x_{-i}) = z_i^* - b \sum_{j \neq i} x_j > 0$*

[6]Strict concavity is actually not required. What is actually required is only the single-peakedness of U with respect to z_i, that is, the strict quasi-concavity of U. Such a single-peakedness assumption is fairly natural and standard (see [Guttman, 1978], see also [Greenberg and Weber, 1993]).

- *If* $b\sum_{j\neq i} x_j \in (z_i^*, \overline{z}_i)$, *then,* $BR_i^{Ind}(x_{-i}) = 0$
- *If* $b\sum_{j\neq i} x_j > \overline{z}_i$ *then,* $BR_i^{Ind}(x_{-i}) = \emptyset$

When $X_i(x_{-i}) = \emptyset$ for $i \in \mathcal{B}$, this simply means that this agent must pay *more* than her own revenue, which is impossible. As a result $BR_i^{Ind}(x_{-i}) = \emptyset$. One may interpret this as an exclusion, similar to the *violation of the no-bankruptcy condition* in aggregative games (see e.g., [Buchholz et al., 2011]). The situation is similar for agents of the group $\mathcal{U}$. When $b\sum_{j\neq i} x_j > \overline{z}_i$ for an agent $i \in \mathcal{U}$, her utility will be *lower* than her reservation utility. As a result $BR_i^{Ind}(x_{-i}) = \emptyset$.

Consider an agent $i \in \mathcal{B}$. Depending upon the willingness-to-pay and the revenue, an agent i may be such that its optimal contribution z_i^* may be equal to r_i or may be lower than r_i. When $z_i^* < r_i$, the budget constraint is not binding and this occurs when z_i^* solves the unconstrained maximization of $U_i(z_i)$. When $z_i^* = r_i$, the constraint is binding. Consider now an agent $j \in \mathcal{U}$ with a high z_j^*. Assume that given x_i with $i \in \mathcal{B}$, the best response of j equal to $BR_j^{Ind}(x_i) := x_j = z_j^* - bx_i$ is higher than $\frac{r_i}{b}$. In such a case, $bx_j > r_i = \overline{z}_i$. From fact 2, $BR_i^{Ind}(x_j) = \emptyset$, that is, no matter the choice of agent i, agent j will always choose a contribution so high that i is left with an empty set. Agent j has a dominant strategy which implies that $BR_i^{Ind}(x_j) = \emptyset$. The following result provides a simple illustration of this when $N = 2$ but nothing is changed in the general case of an arbitrary number N of agents.

Proposition 6 *Let* $b \in (0, 1]$ *and* $J = \{1, 2\}$ *where the group* $\mathcal{B} = \{1\}$ *and the group* $\mathcal{U} = \{2\}$.

(i) A sufficient condition for the non-existence of a Nash equilibrium in individual constraints is $z_2^* > (b + \frac{1}{b})r_1$.

(ii) Assume moreover that $z_1^* = r_1 < z_2^*$ *and that* $z_1^* \geq b\underline{z}_2$ *and let* $\overline{x}_1 := \frac{r_1 - b\underline{z}_2}{(1-b^2)}$. *Then, for any* $\theta \in [0, 1]$, *the pair of strategies* $(x_1^* = \theta\overline{x}_1, x_2^* = \frac{r_1 - \theta\overline{x}_1}{b})$ *is a Nash equilibrium in shared constraint.*

Proof. See the appendix.

When agent 1 faces a budget constraint such that $z_1^* = r_1$, the existence of a Nash equilibrium in individual constraints critically depends upon the heterogeneity (or the dispersion) of the set of ideal contributions $(z_i^*)_{i\in\{1,2\}}$. If this dispersion is too high, that is, if $\frac{z_2^*}{z_1^*}$ is greater than a critical threshold, then, a Nash equilibrium in individual constraints does not exist.

Note that for any $\theta \in [0, 1]$, the pair $(x_1^* = \theta\overline{x}_1, x_2^* = \frac{z_1^* - \theta\overline{x}_1}{b})$ is a Nash equilibrium and is such that $x_1^* + bx_2^* = z_1^*$, it thus follows that for any $\theta \in (0, 1)$, the Nash equilibrium is Pareto optimal.

Corollary 1 *Each Nash equilibrium of the game with shared constraint is Pareto optimal.*

Within our particular model of collective action, when a Nash equilibrium in the game with individual constraints does not exist, there still exists a *continuum* of Nash equilibria in the game with shared constraint that are all Pareto-optimal.

Our analysis also reveals an important property of collective actions. If one only looks at the equilibrium in individual constraints when for instance $z_i^* = r_i$ for each $i \in \mathcal{B}$, the existence of such an equilibrium in individual constraints critically depends upon the dispersion of the ideal contribution $(z_i^*)_{i \in J}$. This thus suggests that for a collective action to be possible (in individual constraints), agents of the group J must have homogenous ideal contributions, something not required for the shared constraint problem. In the general case with $N \geq 2$ agents, let

$$K = \{x \in \mathbb{R}_+^N : x_i + b\sum_{j \neq i} x_j \leq r_i \ \forall i \in \mathcal{B} \text{ and } U_k(x_k + b\sum_{j \neq k} x_j) \geq \overline{u}_k \ \forall k \in \mathcal{U}\}$$

As in proposition 6, as long as the dispersion of $(z_i^*)_{i \in J}$ is "too high", K will be empty and no Nash equilibrium in individual constraints exists. In case in which K is a compact and convex subset of $\mathbb{R}_+^N$, from theorem 1 ([Rosen, 1965]), a Nash equilibrium in the game with shared constraint always exist (since the utility functions are concave) while a Nash equilibrium in individual constraint might not exist.

6 Conclusion

In this paper, we presented the notions of generalized games with individual constraints, generalized games with shared constraint, and generalized games with an endogenous shared constraint generated from individual constraints. We proved a result regarding the existence of Nash equilibria for a generalized game with an endogenous shared constraint generated from individual ones, that is the Nash equilibria of a generalized game with individual constraints are included in the set of Nash equilibria of the generalized game with endogenous shared constraint. We provided a taxinomy of 2×2 generalized games and established two results regarding non-classical games. We then studied different applications of this result, among which a public good problem and an environment control problem.

7 Appendix

Proof of Proposition 1. If $x^* = (x^{*,1}, ..., x^{*,N}) \in E$ is a Nash equilibrium for the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, then:

- $\forall i \in J,\ x_i^* \in X_i(x_{-i}^*)$ so that $x^* \in K$.
- $\forall i \in J,\ \forall x_i \in X_i(x_{-i}^*),\ \theta_i(x_i^*, x_{-i}^*) \leq \theta_i(x_i, x_{-i}^*)$ so $\forall i \in J,\ \forall x_i \in E_i$ such that $(x_i, x_{-i}^*) \in K,\ \theta_i(x_i^*, x_{-i}^*) \leq \theta_i(x_i, x_{-i}^*)$.

It thus follows that if the point x^* is a Nash equilibrium of the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$, it is also a Nash equilibrium for the game with shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J}))$ but the converse is however not true. If the point x^* is a Nash equilibrium of the game with shared constraint generated from the individual constraints $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J}))$, it may be the case that the profile of strategies $x^* \in E$ is not a Nash equilibrium of the game with individual constraints because there may exist $i \in J$ such that the best response $x_i^* := BR_i(x_{-i}^*) \in X_i(x_{-i}^*)$ is such that $x^* \notin K$ where $x^* = (x_i^*, x_{-i}^*)$. Examples in which a profile of strategies which is a Nash equilibrium in shared constraint but not a Nash equilibrium in individual constraints have been given. □

Proof of Proposition 2

Consider the application:

$$\begin{array}{cccc} \theta & : & K & \rightarrow & \mathbb{R} \\ & & x & \mapsto & \sum_{i\in J}\theta_i(x_i) \end{array}$$

Since K is compact and θ is continuous, there exists at least one $x^* \in K$ that minimizes θ, $\theta(x^*) = \min_{x\in K}\theta(x)$. Let us prove that x^* is a Nash equilibrium. If this is was not the case, there would exist an agent $j \in J$ and a strategy $x_j \in E_j$ such that $(x_j, x_{-j}^*) \in K$ and $\theta_j(x_j) < \theta_j(x_j^*)$ so that $\theta_j(x_j) + \sum_{i\neq j}\theta_i(x_i) < \sum_{i\in J}\theta_i(x_i^*)$ and this would contradict the definition of x^*. Therefore x^* is a Nash equilibrium and all Nash equilibria of $(J, E, (\theta_i)_{i\in J}, K)$ are minimizers of θ. As a result, a given Nash equilibrium x^* is Pareto optimal.

□

Proof of Proposition 3

1) Assume that this game has a Nash equilibrium in individual constraints $x^* = (x_1^*, ..., x_N^*)$. Such a Nash equilibrium must satisfy $\sum_j a_{ij}x_j^* = S_i$ for all i. Indeed, assume that for a given i we have $\sum_j a_{ij}x_j^* < S_i$, then agent i could still increase its payoff $\theta_i(x_i)$ with a x_i higher than x_i^* still satisfying the constraint, and therefore x^* would not be a Nash equilibrium. Therefore a Nash equilibrium $x^* = (x_1^*, ..., x_N^*)$ for this problem with individual constraints must satisfy: $AX^* = S$ with $X^* \in \mathbb{R}_+^N$.

And this has a solution if and only if the linear system $Ax = S$ has a solution $x^* \in \mathbb{R}_+^N$.

- If A is invertible, we have an explicit formula for the Nash equilibrium candidate vector $x^* = A^{-1}S$

- If A is not invertible and the linear system $Ax = S$ admits one solution $x^{*,0} \in \mathbb{R}_+^N$, then $Ax = Ax^{*,0}$, which is equivalent to $A(x - x^{*,0}) = 0$, which is equivalent to $x \in \{x^{*,0} + y, y \in \ker(A)\} \cap \mathbb{R}_+^N$ and there are infinitely many Nash equilibria.

- If the linear system $Ax = S$ admits no solution in $\mathbb{R}_+^N$, then there is no Nash equilibrium.

2) We provide two different proofs of this result:

- First proof :

  It is easy to see that the game always satisfies the assumptions of the existence result of Rosen for n-person concave games (see theorem 1 in the text). Indeed, the payoff functions $\theta_i(x_i)$ are continuous and concave, and the shared constraint space K is clearly a convex compact space:

  $K = \{x \in \mathbb{R}_+^N : \sum_{j\in J} a_{ij}x_j \leq S_i, i = 1, 2, ..., N\}$

  Therefore, there always exist a Nash equilibrium for the game with shared constraint.

- Second proof :

  This is a non-classical game, K is a compact set and the cost functions are continuous, therefore this game satisfies the assumptions of Proposition 2, and a Nash equilibrium in shared constraint always exist. Moreover, $\mathcal{N}_{Shared}$ coincides exactly with the minimizers of the total cost function $\sum_{i\in J} \theta_i(x_i)$ on K and all its elements are Pareto optimal.

□

Proof of proposition 4.

We shall give here a result more detailed than the one stated in the text. In what follows, we actually offer a complete characterization of the existence or non-existence of the Nash equilibrium in individual strategies. The environmental problem formulated as a game with individual constraints has a Nash equilibrium if and only if the linear system $A^T \text{diag}(c)x = S$ (A^T denotes the transpose of A) admits at least one solution $x^* \in \mathbb{R}_+^N$.

PropositionA 4

- If the linear system $A^T \text{diag}(c)x = S$ admits no solution in $\mathbb{R}_+^N$, then there is no Nash equilibrium.

- If the matrix A is invertible, then, there exists a unique solution x^* to the linear system, and if x^* is in $\mathbb{R}_+^N$, there exists a unique Nash equilibrium of the form:

$$x^* = \left(x_1^* = \frac{1}{c_1} \sum_{j=1}^{N} b_{1j} S_j, ..., x_i^* = \frac{1}{c_i} \sum_{j=1}^{N} b_{ij} S_j, ..., x_N^* = \frac{1}{c_N} \sum_{j=1}^{N} b_{Nj} S_j \right)$$

  with $(B_{i,j})_{i,j\in J} = (A^T)^{-1}$. If x^* is not in $\mathbb{R}_+^N$, there is no Nash equilibrium.

- If the matrix A is not invertible and the linear system $A^T\text{diag}(c)x = S$ admits at least one solution $x^{*,0} \in \mathbb{R}_+^N$, then the linear system admits infinitely many solutions, and there are infinitely many Nash equilibria given by the set $\{x^{*,0} + y, y \in \ker(A^T\text{diag}(c))\} \cap \mathbb{R}_+^N$

Proof of propositionA 4

Let's assume that this game has a Nash equilibrium $x^* = (x_1^*, ..., x_N^*)$. Such a Nash equilibrium must satisfy $\sum_j a_{ji} \times c_j \times x_j^* = S_i$ for all i. Indeed, assume that for a given i we have $\sum_j a_{ji} \times c_j \times x_j^* < S_i$, then country i could still increase its profit $f_i(x_i)$ with a x_i higher than x_i^* still satisfying the constraint, and therefore x^* would not be a Nash equilibrium. Therefore a Nash equilibrium $x^* = (x_1^*, ..., x_N^*)$ for this problem with individual constraints must satisfy: $A^T\text{diag}(c)x^* = S$.

And this has a solution if and only if the linear system $A^T\text{diag}(c)x = S$ has a solution $x^* \in \mathbb{R}_+^N$.

- If A is invertible, we have an explicit formula for the Nash equilibrium candidate vector since:

$$\text{diag}(c)x^* = (\text{A}^T)^{-1} \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix}$$

If we denote $B = (A^T)^{-1}$, we have:

$$\text{diag}(c)x^* = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1N} \\ b_{21} & b_{22} & ... & b_{2N} \\ ... & ... & ... & ... \\ b_{N1} & b_{N2} & ... & b_{NN} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix}$$

And for all i $x_i^* = \frac{1}{c_i}\sum_{j=1}^N b_{ji}S_j$, and we have a unique Nash equilibrium candidate $x^* = (x_1^* = \frac{1}{c_1}\sum_{j=1}^N b_{1j}S_j, ..., x_i^* = \frac{1}{c_i}\sum_{j=1}^N b_{ij}S_j, ..., x_N^* = \frac{1}{c_N}\sum_{j=1}^N b_{Nj}S_j)$.

- If A is not invertible and the linear system $A^T\text{diag}(c)x = S$ admits one solution $X^{*,0} \in \mathbb{R}_+^N$, then $A^T\text{diag}(c)x = A^T\text{diag}(c)x^{*,0}$, which is equivalent to $A^T\text{diag}(c)(x - x^{*,0}) = 0$, which is equivalent to $x \in \{x^{*,0} + y, y \in \ker(A^T\text{diag}(c))\} \cap \mathbb{R}_+^N$ and there are infinitely many Nash equilibria.

- If the linear system $A^T\text{diag}(c)x = S$ admits no solution in $\mathbb{R}_+^N$, then there is no Nash equilibrium.

□

Proof of proposition 6

Part i). Assume that $x_1 = r_1$. The best response of 2, x_2, is equal to $z_2^* - br_1$. As seen in the paragraph before stating Proposition 6, if $z_2^* - br_1 > \frac{r_1}{b}$, then, $x_2 > r_1$ so that the best response of agent 1 is empty. Solving $z_2^* - br_1 > \frac{r_1}{b}$ yields part (i) □

Part ii). Let $K = \{(x_1,x_2) \in \mathbb{R}_2^+ : x_1 + bx_2 \le r_1$ and $U_2(x_1,x_2) \ge \overline{u}_2\}$. Let us denote BR_i^{Sh} the best response of agent i in shared constraint. For a Nash equilibrium with shared constraint to exist, we must first prove the non-vacuity of K, which in turn is equivalent to the existence of a solution that solves the following system.

$$x_1 + bx_2 \le r_1 \tag{13}$$

$$U_2(x_2 + bx_1) \ge \overline{u}_2 \tag{14}$$

We know for agent 2 that there exists two critical thresholds $\underline{z}_2$ and $\overline{z}_2$ defined as $\underline{z}_2 = \inf_{z_2} U_2(z_2) = \overline{u}_2$ and $\underline{z}_2 = \sup_{z_2} U_2(z_2) = \overline{u}_2$. It thus follows that equation (14) is equivalent to $z_2 \in [\underline{z}_2, \overline{z}_2]$, where $z_2 = x_2 + bx_1$.

For K to be non-empty, it thus suffices to find a solution to the following system

$$x_1 + bx_2 \le r_1 \tag{15}$$

$$\underline{z}_2 \le x_2 + bx_1 \le \overline{z}_2 \tag{16}$$

First let us prove that (16) reduces to $\underline{z}_2 \le x_2 + bx_1 \le z_2^*$. Indeed, if we had $x_2 + bx_1 > z_2^*$, then this would imply $x_2 + bx_1 > (b + \frac{1}{b})r_1$, which would imply $bx_2 > r_1 = \overline{z}_1$, which would imply $BR_1^{Ind}(x_2) = \emptyset$. So inequation (16) becomes

$$\underline{z}_2 \le x_2 + bx_1 \le z_2^* \tag{17}$$

Solving inequation (15) in x_2 yields $x_2 \le \frac{r_1 - x_1}{b}$ while solving inequation (17) yields $x_2 \ge \underline{z}_2 - bx_1$. For a solution in x_2 to exist, $\frac{r_1 - x_1}{b}$ must be higher than $\underline{z}_2 - bx_1$. Solving $\frac{r_1 - x_1}{b} \ge \underline{z}_2 - bx_1$ yields $r_1 - b\underline{z}_2 \ge x_1(1 - b^2)$. Then a necessary condition is $r_1 - b\underline{z}_2 \ge 0$.

Assuming that $r_1 - b\underline{z}_2 \ge 0$, x_1 must be in $\left[0, \frac{r_1 - b\underline{z}_2}{(1-b^2)}\right]$. Let $\overline{x}_1 := \frac{r_1 - b\underline{z}_2}{(1-b^2)}$ be the maximal value of x_1 and assume that $x_1 = \theta\overline{x}_1$ for some $\theta \in [0,1]$. Consider x_2^* such that $\theta\overline{x}_1 + bx_2^* = r_1 = z_1^*$, which means that $x_2^* = \frac{r_1 - \theta\overline{x}_1}{b}$. Given $x_2^* = \frac{r_1 - \theta\overline{x}_1}{b}$, agent 1 will choose x_1 such that $x_1 + bx_2^* = z_1^*$ so that $BR_1^{Sh}(\frac{r_1 - \theta\overline{x}_1}{b}) = x_1^* = \theta\overline{x}_1$.

Let us prove that $x_2^* = BR_2^{Sh}(\theta\overline{x}_1)$. Given $\theta\overline{x}_1$, agent 2 is seeking to maximize x_2 subject to the two constraints (15) and (17). x_2^* is the maximum solution with (15). If the best response in shared constraint $BR_2^{Sh}(\theta\overline{x}_1)$ of agent 2 was higher than x_2^* then this would not satisfy the constraint (15).

Now, let us prove that $\underline{z}_2 < x_2^* + b\theta\overline{x}_1 < z_2^*$.

Note that $x_2^* + b\theta\overline{x}_1 = \frac{r_1 - \theta\overline{x}_1}{b} + b\theta\overline{x}_1 := h(\theta)$. The function $h(\theta)$ reaches its maximum when $\theta = 0$ and its maximum value is equal to $\frac{r_1}{b}$, which is lower than z_2^*. It reaches its minimum value when $\theta = 1$ and the minimum value is equal to $\frac{r_1 - \overline{x}_1}{b} + b\overline{x}_1 = \frac{r_1}{b} + \frac{b^2-1}{b}\overline{x}_1 = \frac{r_1}{b} + \frac{b^2-1}{b}\frac{r_1 - b\underline{z}_2}{1-b^2} = \frac{r_1}{b} - \frac{r_1}{b} + \underline{z}_2 = \underline{z}_2$. Therefore $\underline{z}_2 < x_2^* + b\theta\overline{x}_1 < z_2^*$ is always satisfied so $BR_2^{Sh}(\theta\overline{x}_1) = x_2^*$. It thus follows that for all $\theta \in [0,1]$, the pair of strategies $(x_1^* = \theta\overline{x}_1, x_2^* = \frac{r_1 - \theta\overline{x}_1}{b})$ is a Nash equilibrium in shared constraint. □

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