A Generalized Nash Equilibrium Problem arising in banking regulation: An existence result with Tarski's theorem

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February 25, 2022

Abstract

When hit with an adverse shock, banks that do not comply with capital regulation sell risky assets to satisfy their solvency constraint. When financial markets are imperfectly competitive (oligopoly situation), this naturally gives rise to a GNEP. In [Braouezec and Wagalath, 2019], they consider the single risky asset framework and show that, using Tarski's theorem, a Nash equilibrium exists. We extend this framework to an arbitrary number of assets and show that, when markets are sufficiently competitive, Tarski's theorem can still be used. As a byproduct, we show that Tarski's theorem can be used to solve a GNEP with individual constraints.

Keywords: Generalized games, banking regulation, Cournot oligopoly, asset sales, Tarki's fixed point theorem

JEL Classification: G21, G28, D43

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1 Introduction

Compared to other financial institutions such as hedge funds, banks are regulated; their main capital ratio called the risk-based capital ratio (RBC) defined as regulatory capital or equity E divided by the risk-weighted assets RWA, that is, $\frac{E}{\text{RWA}}$, must be greater than a critical threshold around 10% at all times[1]. After an adverse systemic shock, as e.g., in mid-September 2008 when Lehman Brothers failed, some banks may fail to satisfy their capital requirement. To comply with regulation, the quickest solution consists in selling a portion of their risky assets to decrease their risk-weighted assets. However, when many banks sell the same asset at the same time, something called a generalized asset shrinkage ([Hanson et al., 2011]), the price of the asset will decrease through the market mechanism and will further deplete the capital of the bank, that is, this generates a kind of death spiral (e.g., [Brunnermeier and Pedersen, 2009], [Hanson et al., 2011]). It is usual to call such a phenomenon a price-mediated contagion problem (e.g., [Feinstein, 2020]), as opposed to direct contagion which is related to the network of exposures (e.g., [Feinstein, 2017], see [Glasserman and Young, 2016] for a review). In [Braouezec and Wagalath, 2019], they consider a game theoretic price-mediated contagion model in the particular case of one risky asset and show, using Tarski's theorem, that at least one Nash equilibrium exists. They however fail to recognize that this setting is indeed a generalized game. In this paper, we reconsider the [Braouezec and Wagalath, 2019] setting in the general case of $n \geq 2$ assets and prove, under some conditions, the existence of a Nash equilibrium using Tarski's theorem. This theorem is widely applied in financial network models (e.g., [Glasserman and Young, 2016]) but not in generalized games (see [Dutang, 2013] for a review of existence theorems, see also [Arrow and Debreu, 1954] and [Facchinei and Kanzow, 2010]).

The aim of this paper is twofold. First, it is to show that due to both banking regulation and the market mechanism, a generalized game naturally occurs when one considers the asset sale problem between banks. Second, it is to show how Tarski's theorem can be used to prove the existence of a Nash equilibrium in a generalized game. As opposed to many generalized games papers (e.g., [Aussel and Dutta, 2008] [Facchinei et al., 2007]), we do not make use of the variational inequality approach nor Ichiishi's theorem (see [Ichiishi, 1983], see also [Dutang, 2013]).

2 The generalized game

We consider the extension of [Braouezec and Wagalath, 2019] with a finite number of risky assets. Let $\mathcal{B} = \{1, ..., p\}$ be the set of banks and $\mathcal{S} = \{1, ..., n\}$ be the set of risky securities. Each bank i holds a quantity q_{ij} of security j (stocks, bonds, index...) for which the price P_j has been hit with a shock Δ_j in percentage at time $t = 0$, i.e., it is equal to $P_j(1 - \Delta_j)$. Right after the shock $\Delta = (\Delta_1, ..., \Delta_n)$, the total value of the assets of each bank i is equal to $A_i(\Delta) := \sum_{j=1}^{n} P_j(1 - \Delta_j) q_{ij} = A_i - \sum_{j=1}^{n} q_{ij} P_j \Delta_j$ where $\sum_{j=1}^{n} q_{ij} P_j \Delta_j$ is the *loss* incurred by the bank. The balance sheet of bank i is given below where A_i is the total assets before the shock.

Balance-sheet of bank i

Assets	Liabilities
$\text{A}_{i1} - q_{i1} P_1 \Delta_1$	Debt: D_i
$\vdots$	
$\text{A}_{in} - q_{in} P_n \Delta_n$	Equity: $E_i(\Delta)$
$A_i - \sum_{j=1}^{n} q_{ij} P_j \Delta_j$	$E_i(\Delta) + D_i$

[1] For a short introduction, see [BCBS, 2017] publicly available on the BIS website.

As long as the the bank is solvent, the total value of the assets is equal to the total value of the liabilities (i.e., $A_i(\Delta) = E_i(\Delta) + D_i$) so that the capital can absorb the loss, that is, $E_i(\Delta) = E_i - \sum_{j=1}^{n} q_{ij} P_j \Delta_j > 0$ where E_i is the capital before the shock. The positivity of the capital is however not enough. Banking regulation imposes a risk-based capital ratio for each bank i to be greater than $\theta_{i,min}$ (in practice $\theta_{i,min} \approx 10\%$). Let $\alpha_{ij} \in [0, 1]$ be the regulatory weight[2] of asset j for bank i.

Assumption 1 *For each $i \in \mathcal{B}$*

$$\alpha_{i,1} > \alpha_{i,2} > ... > \alpha_{i,n} \tag{1}$$

The regulatory weights can be directly provided by regulators to banks (standardised approach) so that they are identical for each bank. But they can also be computed by the bank (internal model approach), which means that two different banks may have different estimation of the weight of a given asset. However, in the post-subprime crisis regulation called Basel III, banks have much less freedom than before to make use of internal models. Assumption 1 thus is consistent with the actual banking regulation[3] in that, when banks use internal models, some heterogeneity in the weight estimation is allowed but restricted. The risk-weighted assets are equal to $\text{RWA}_i(\Delta) := \sum_j \alpha_{ij} P_j (1 - \Delta_j) q_{ij}$. For the sake of financial interest, we assume that each bank must react after the shock.

Assumption 2 *For each $i \in \mathcal{B}$, $E_i(\Delta) > 0$ but*

$$\theta_i(\Delta) := \frac{E_i(\Delta)}{\text{RWA}_i(\Delta)} < \theta_{i,min} \tag{2}$$

To restore their capital ratio back above $\theta_{i,min}$, banks are assumed to sell a portion $x_{ij} \in [0, 1]$ of security $j = 1, ..., n$. Let $\sum_{k\in\mathcal{B}} x_{kj} q_{kj}$ be the total quantity of security j sold by banks. When markets are imperfectly competitive, the price is impacted by such sales. For simplicity, we consider a linear price impact model (e.g., [Braouezec and Wagalath, 2019]) for which the price of security j after the asset sale is equal to

$$P_j^{\text{after}} = P_j^{\text{before}} \times \left(1 - \frac{\sum_{i\in\mathcal{B}} x_{kj} q_{kj}}{\Phi_j}\right) \tag{3}$$

where $\Phi_j >> \sum_{k\in\mathcal{B}} q_{kj}$ is called the market depth that measures the competitiveness of market j. The greater Φ_j, the more competitive the market of security j. At the limit, when Φ_j is infinite, it is perfectly competitive. Let $x_i \in [0, 1]^n$ be the liquidation of bank i and $x \in [0, 1]^{np}$ be a of liquidation of the set of banks. Let

$$L_i(x_i) := \sum_{j=1}^{n} x_{ij} q_{ij} P_j (1 - \Delta_j) \tag{4}$$

be the total value of the assets sold by bank i. It is not difficult to show that the risk-based capital ratio of bank i is equal to

$$\theta_i(\Delta, x) := \frac{E_i(\Delta, x)}{\text{RWA}_i(\Delta, x)} = \frac{E_i - \sum_{j=1}^{n} q_{ij} P_j \times (\Delta_j + \frac{\sum_{k\in\mathcal{B}} x_{kj} q_{kj}}{\Phi_j}(1 - \Delta_j))}{\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j) \left(1 - \frac{\sum_{k\in\mathcal{B}} x_{kj} q_{jk}}{\Phi_j}\right)(1 - x_{ij})} \tag{5}$$

[2] Some weights may be higher than one. For simplicity, we assume that they are between zero and one

[3] See once again [BCBS, 2017]. It should however be noted that all the results hold without this assumption. It suffices to define suitable permutations.

As usual in game theory, let $x = (x_i, x_{-i})$. Since the capital ratio of bank i depends upon x_i but also upon $x_{-i} \in [0,1]^{n(p-1)}$, what banks $k \neq i$ are liquidating; the asset sale problem is *strategic*. Let

$$X_i(x_{-i}) = \{x_i \in [0,1]^n, \ \theta_i(x_i, x_{-i}) \geq \theta_{i,min}\} \tag{6}$$

be the solvency constraint of a given bank i. At time $t = 0$, each bank i chooses its liquidation strategy $x_i := (x_{ij})_{j \in \mathcal{S}} \in [0,1]^n$ and prices after liquidation are disclosed, i.e., $P^{\text{after}} = (P_1^{\text{after}}, ..., P_n^{\text{after}})$ at time $t = 1$. Each bank $i \in \mathcal{B}$ is assumed to solve the following constrained optimization problem.

$$\min_{x_i} L_i(x_i) := \sum_{j=1}^{n} x_{ij} q_{ij} P_j (1 - \Delta_j) \quad s.t. \quad x_i \in X_i(x_{-i}) \tag{7}$$

As opposed to many game theoretic framework envisaged in economics and finance, interaction between banks takes place through the solvency constraint but not through the objective function, that is, (7) defines a particular case of a *generalized game*[4]. Let $\Phi := (\Phi_1, ..., \Phi_n)$ and let

$$K_{\Delta,\Phi} = K := \{x \in [0,1]^{np} : \forall i \in \mathcal{B}, x_i \in X_i(x_{-i})\} \subset [0,1]^{np} \tag{8}$$

be the set admissible strategies of the generalized game. We now recall the definition of a Nash equilibrium for our game.

Definition 1 *The profile of strategies $x^* \in K$ is a Nash equilibrium of the asset sale game if, for each $i \in \mathcal{B}$ and each $x_i \in [0,1]^n$ such that $x_i \in X_i(x^*_{-i})$, it holds true that $L_i(x^*_i, x^*_{-i}) \leq L_i(x_i, x^*_{-i})$.*

3 Perfectly competitive markets

When markets are perfectly competitive, $\frac{1}{\Phi_j} = 0$ for all j. From equation (5), the risk-based capital ratio of bank i reduces to

$$\theta_i(\Delta, x_i) = \frac{E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j}{\sum_{j=1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j)(1 - x_{ij})} \tag{9}$$

so that the decision problem is not anymore strategic; the capital ratio only depends upon x_i.

Lemma 1 *The risk-based capital ratio $\theta_i(\Delta, x_i)$ an increasing function of x_{ij} for each $i \in \mathcal{B}$ and each $j \in \mathcal{S}$.*

Proof. Since the numerator of $\theta_i(\Delta, x_i)$ is invariant with respect to x_{ij} while the denominator decreases with x_{ij}, the result follows $\square$

The next proposition provides a characterization of the optimal liquidation strategy. Note that the optimization problem reduces to a linear programming problem. A proof in the case in which $n = 2$ can be found in [Braouezec and Wagalath, 2018].

Proposition 1 ***(Characterization of the optimal strategy)*** *In addition to equation (1), assume that Δ is such that for each bank $i \in \mathcal{B}$, $\theta_i(\Delta) \in (0, \theta_{i,min})$. When markets are perfectly competitive, there is a unique optimal liquidation vector $(x^*_{i,1}, ..., x^*_{i,2}, ..., x^*_{i,n}) \in [0,1]^n \backslash \{(1,1,...,1)\}$ of the form $(1, ..., 1, x^*_{i,h}, 0, ...0)$ where $x^*_{i,h} \in (0,1)$ for some integer $h \in \{1, ..., n\}$ is such that*

$$x^*_{i,h} = \frac{1}{\alpha_{i,h} q_{i,h} P_h (1 - \Delta_h)} \left[\sum_{j=h+1}^{n} \alpha_{ij} q_{ij} P_j (1 - \Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^{n} q_{ij} P_j \Delta_j) \right] \tag{10}$$

[4] In a generalized game, interaction occurs both through the constraint and the objective function.

Proof. The denominator of (9) tends to zero when x_{ij} tends to one for all j so that the capital ratio that tends to infinity. Since $\theta_{i,min} < \infty$, a solution exists. For $x_i^* = (x_{i1}^*, ..., x_{in}^*)$, the constraint is clearly binding; $\theta_{i,t+1}(\Delta, x_i) = \theta_{i,min}$, that is,

$$\frac{E_{i,t} - \sum_{j=1}^n q_{ij} P_j \Delta_j}{\sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j)(1 - x_{ij}^*)} = \theta_{i,min} \tag{11}$$

which implies that:

$$\sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j) x_{ij}^* = \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^n q_{ij} P_j \Delta_j)$$

Letting $X_{ij} = q_{ij} P_j (1 - \Delta_j) x_{ij}^*$, and $C_i = \sum_{j=1}^n \alpha_{ij} q_{ij} P_j (1 - \Delta_j) - \frac{1}{\theta_{i,min}} (E_{i,t} - \sum_{j=1}^n q_{ij} P_j \Delta_j)$, equation (11) is equivalent to:

$$\sum_{j=1}^n \alpha_{ij} X_{ij} = C_i \tag{12}$$

Each bank i seeks to minimize $\sum_{j=1}^n X_{ij}$. Therefore, the unique solution consists in selling asset 1 with the highest risk weight α_{i1}, then asset 2 to asset $h \leq n$ until the capital ratio is restored □

Proposition 1 says that it is optimal for each bank i to first sell the asset with the highest regulatory weight. If this is not enough to restore the capital ratio, it is now optimal to sell 100% of asset 1 and a portion of risky asset 2. If this is not enough, it is optimal to sell 100% of asset 1 and 2 and a portion of risky asset 3 and so on and so forth. The optimal liquidation strategy follows the order of the weights.

4 Imperfectly competitive markets: existence result with Tarski's theorem

4.1 Preliminary results and main assumptions

We shall prove few preliminary results.

Assumption 3 *K is never empty, that is,*

$$\forall i \in \mathcal{B} \;\; \forall x \in [0,1]^{np}, E_i(\Delta, x) > 0 \tag{13}$$

Lemma 2 *Under assumption 3, regardless of $x_{-i} \in [0,1]^{n(p-1)}$, $X_i(x_{-i}) \neq \emptyset$*

Proof. From assumption 3, $E_i(\Delta, 1) > 0$ where 1 is the np-dimensional vector. Let $E_i(\Delta, 1) := \xi_i > 0$ and note that $E_i(\Delta, x_i, x_{-i}) \geq \xi_i$ regardless of $x_{-i} \in [0,1]^{n(p-1)}$. Since for each i, $\text{RWA}_i(x_i)$ tends to zero when x_i tend to the n-dimensional vector 1, $\lim_{x_i \to 1} \frac{\xi_i}{\text{RWA}_i(x_i)} \to \infty$. Since $\theta_i(\Delta, x_i, x_{-i}) \geq \frac{\xi_i}{\text{RWA}_i(x_i)}$ regardless of $x_{-i} \in [0,1]^{n(p-1)}$ and since $\theta_{i,min} < \infty$, there exists $x_i \in [0,1]^n \backslash \{1, ..., 1\}$ such that $\theta_i(\Delta, x_i, x_{-i}) = \theta_{i,min}$, hence $X_i(x_{-i}) \neq \emptyset$ □

If there is a Nash equilibrium to the generalized game, each bank will be solvent at equilibrium, that is, fo reach i, $x_i \in [0,1]^n \backslash \{1, ..., 1\}$. For notational simplicity, we may denote $\theta_i(\Delta, x_i, x_{-i})$ by $\theta_i(.)$.

Lemma 3 *Given x_i, $\theta_i(.)$ is a decreasing function of x_{kj} for all $k \neq i$ and all $j = 1, ..., n$. In particular, $\theta_i(\Delta, x_i, x_{-i})$ is a decreasing function of x_{-i}.*

Proof. Let $N(x) := E_i(\Delta, x)$ be the numerator of the risk-based capital ratio and $D(x)$ be its denominator (see equation (5)). Consider $\frac{\partial \theta_i}{\partial x_{kj}}$ when $k \neq i$:

$$\frac{\partial \theta_i(.)}{\partial x_{kj}} = \frac{q_{ij} P_j q_{kj}(1-\Delta_j)}{\Phi_j D(x)^2}[\alpha_{ij}(1-x_{ij})N(x) - D(x)]$$

which has same sign as $\alpha_{ij}(1-x_{ij})N(x) - D(x)$. We notice that $\alpha_{ij}(1-x_{ij}) \leq 1$, so if $N(x) < D(x)$, we have that $\frac{\partial \theta_i}{\partial x_{kj}} < 0$. Since $\theta_{i,min} << 1$, it is always true that $N(x) < D(x)$. Therefore, if $x_{-i} \leq y_{-i}$, we have that $\theta_i(x_i, x_{-i}) \geq \theta_i(x_i, y_{-i})$ □

Lemma 4 *When the market depths Φ are large enough, $\theta_i(.)$ is an increasing function of x_{ij} for each $j = 1, 2, ..., n$.*

Proof. Consider $\frac{\partial \theta_i}{\partial x_{kj}}$ when $k = i$:

$$\frac{\partial \theta_i}{\partial x_{ij}} = \frac{q_{ij} P_j (1-\Delta_j)}{D(x)^2}\left[\alpha_{ij} N(x)[1 + \frac{q_{ij}}{\Phi_j}(1-x_{ij}) - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x)\right] \tag{14}$$

which has same sign as $\alpha_{ij} N(x)[1 + \frac{q_{ij}}{\Phi_j}(1-x_{ij}) - \frac{\sum_{k \in B} x_{kj} q_{kj}}{\Phi_j}] - \frac{q_{ij}}{\Phi_j} D(x)$. From equation (13), $N(x) > 0$ for all x. From equation (14), it thus follows that when market depths are high enough, $\frac{\partial \theta_i(.)}{\partial x_{ij}} > 0$ □

Corollary 1 *There exists a critical smallest market depths vector $\Phi' := (\Phi'_1, ..., \Phi'_n)$ such that, regardless of $x_{-i} \in [0,1]^{n(p-1)}$, for each $i \in \mathcal{B}$ and each $j \in \mathcal{S}$, $\theta_i(.)$ is an increasing function of $x_{ij} \in [0,1]$.*

Assumption 4 *The market depths satisfy $\Phi \geq \Phi'$.*

Note that assumption 3 does not imply assumption 4. If given market depth Φ_j is not high enough, for at least one bank i, $\theta_i(.)$ may not be an increasing function of x_{ij} yet for each x, $E_i(\Delta, x) > 0$.

4.2 Main result

Before proving the main result, we show that when the market depths are high enough, the optimal liquidation strategy is identical to the one found in Proposition 1.

Lemma 5 *There exists a critical market depths vector $\Phi^0 := (\Phi^0_1, ..., \Phi^0_n)$ such that, for all $\Phi \geq \Phi^0$, the optimal liquidation strategy of each bank $i \in \mathcal{B}$ is identical to the one in Proposition 1.*

Proof. See the appendix.

Lemma 6 *When $\Phi \geq \max\{\Phi', \Phi^0\}$, $BR_i(x_{-i})$ is an non-decreasing function of x_{-i}.*

Proof. Let $x \in [0,1]^{np}$ and $y \in [0,1]^{np}$ such that $x \leq y$ so that $x_{-i} \leq y_{-i}$. When $\Phi \geq \Phi^0$, the best responses are given as in lemma 5. Let $BR_i(x_{-i}) = (1, ..., 1, x_{ih}, 0, ...0)$ and $BR_i(y_{-i}) = (1, ..., 1, y_{il}, 0, ..., 0)$. Since $\theta_i(BR_i(x_{-i}), x_{-i}) = \theta_i(BR_i(y_{-i}), y_{-i}) = \theta_{i,min}$, using the properties of the best responses, $\theta_i((1, ..., 1, x_{ih}, 0, ..., 0), x_{-i}) = \theta_i((1, ..., 1, y_{il}, 0, ..., 0), y_{-i}) = \theta_{i,min}$. Since $x_{-i} \leq y_{-i}$, from Lemma 3 we have that $\theta_{i,min} = \theta_i((1, ..., 1, x_{ih}, 0, ...0), x_{-i}) \geq \theta_i((1, ..., 1, x_{ih}, 0, ...0), y_{-i})$

and therefore $\theta_{i,min} = \theta_i((1,...,1,y_{il},0,...,0),y_{-i}) \geq \theta_i((1,...,1,x_{ih},0,...,0),y_{-i})$. From Lemma 4, when $\Phi \geq \Phi'$, since given z_{-i}, $\theta_i(z_i,z_{-i})$ is an increasing function of z_{ij} for all $j \in [1,..,n]$, we necessarily have that $l \geq h$ and in the case that $l = h$ we have that $y_{il} \geq x_{il}$. Therefore $BR_i(x_{-i}) = (1,...,1,x_{ih},0,...,0) \leq (1,...,1,y_{il},0,...,0) = BR_i(y_{-i})$, which concludes the proof □

If the market depths are not high enough, then, there exists $x \leq y$ such that given x_{-i}, $\alpha_{i1}(1 - \delta_1(\Phi_1,x)) > \alpha_{i2}(1-\delta_2(\Phi_2,x))$ for all x_i while given y_{-i}, $\alpha_{i2}(1-\delta_2(\Phi'_2,y)) > \alpha_{i1}(1-\delta_1(\Phi'_1,y))$ for all y_i. In such a case, the best responses are no more increasing.

Tarski's theorem ([Tarski, 1955], see also [Vives, 1990]). *Let $(L,\geq)$ be a complete lattice and f a non decreasing function from L to L and $\mathcal{F}$ the set of fixed points of f. Then, $\mathcal{F}$ is non-empty and $(\mathcal{F},\geq)$ is a complete lattice. In particular, $\sup_x \mathcal{F}$ and $\inf_x \mathcal{F}$ belong to $\mathcal{F}$.*

Let $([0,1]^{np},\leq)$ be a lattice with $\leq$ defined by the natural order $x \leq y \iff x_i \leq y_i$ for each $i = 1,...,p$ where $x_i \leq y_i$ component-wise. Note that $[0,1]^{np}$ is the product of compact intervals and thus is a complete lattice. Consider a function f from $([0,1]^{np},\leq)$ to $([0,1]^{np},\leq)$. We shall now apply Tarski's theorem to the function f

$$f(x) = (BR_1(x_{-1}),...,BR_p(x_{-p})) \tag{15}$$

and shall show that it is a non-decreasing function from $([0,1]^{np},\leq)$ to $([0,1]^{np},\leq)$.

Proposition 2 *When $\Phi \geq \max\{\Phi',\Phi^0\}$, there exists a smallest Nash equilibrium $x^* = (x_1^*,...,x_p^*) \in ([0,1]^n \backslash \{(1,1,...,1)\})^p$ to the generalized game defined in (7).*

Proof of proposition 2

Let us consider $x \in [0,1]^{np}$ and $y \in [0,1]^{np}$ such that $x \leq y$. Therefore, $x_{-i} \leq y_{-i}$ so for all $z_i \in [0,1]^n$ we have that $\theta_i(z_i,x_{-i}) \geq \theta_i(z_i,y_{-i})$. This implies that $X_i(y_{-i}) = \{z_i \in [0,1]^n; \theta_i(z_i,y_{-i}) \geq \theta_{i,min}\} \subset X_i(x_{-i}) = \{z_i \in [0,1]^n; \theta_i(z_i,x_{-i}) \geq \theta_{i,min}\}$, and by assumption these two sets are not empty. Therefore, from Lemma 6 $BR_i(x_{-i}) \leq BR_i(y_{-i})$ for all i. Hence, f is a non-decreasing function from $([0,1]^{np},\leq)$ to $([0,1]^{np},\leq)$, and therefore it satisfies the assumptions of Tarski's theorem. As a consequence, the set of fixed points of f is not empty, so that there exists at least one Nash equilibrium. If there are multiple Nash equilibria, since there ordered, by Tarski's theorem, $\inf_x \mathcal{F}$ belong to $\mathcal{F}$, the set of Nash equilibrium and $x^* = \inf_x \mathcal{F}$. At equilibrium, each bank i is solvent so that it must be the case that $x_i^* \in [0,1]^n \backslash \{(1,1,...,1)\}$ □

Interestingly, Tarski's theorem allows us to prove the existence of a Nash equilibrium within our generalized game. To the best of our knowledge, this is the first application of Tarski's theorem to a generalized game under individual constraints.

5 Conclusion

We have shown how a generalized game naturally occurs between regulated banks when markets are imperfectly competitive. When they are however sufficiently competitive, Tarski's theorem can be used to prove the existence of a Nash equilibrium. Many things remain however to be done. In particular, does there exist a Nash equilibrium when markets are not enough competitive?

6 Proofs

Proof of Lemma 5 Given x_{-i}, the best response $(x_{i,1},...,x_{i,n})$ is such that the constraint is binding, that is,

$$\frac{E_i - \sum_{j=1}^n q_{ij}P_j(\Delta_j + \frac{\sum_{k\in\mathcal{B}} x_{kj}q_{kj}}{\Phi_j}(1-\Delta_j))}{\sum_{j=1}^n \alpha_{ij}q_{ij}P_j\,(1-\Delta_j)\left(1-\frac{\sum_{k\in\mathcal{B}} x_{kj}q_{jk}}{\Phi_j}\right)(1-x_{ij})} = \theta_{i,min} \tag{16}$$

We shall now show that, by suitably relabeling the quantities, equation (16) can be written, up to functions in $\frac{1}{\Phi_j}$, as (12). Define $\epsilon_j(\Phi_j,x) := \frac{\sum_{k\in\mathcal{B}} x_{kj}q_{kj}}{\Phi_j}$. From eq (16), we have

$$\sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)(1-\epsilon_j(\Phi_j,x))x_{ij} =$$
$$\sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)(1-\epsilon_j(\Phi_j,x)) - \frac{E_i - \sum_{j=1}^n q_{ij}P_j(\Delta_j+\epsilon(\Phi_j,x))(1-\Delta_j))}{\theta_{i,min}}$$

Let $X_{ij} = q_{ij}P_j(1-\Delta_j)x_{ij}$ and $C_i = \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j) - \frac{1}{\theta_{i,min}}(E_{i,t} - \sum_{j=1}^n q_{ij}P_j\Delta_j)$. Equation (16) is therefore equivalent to:

$$\sum_{j=1}^n \alpha_{ij}(1-\epsilon_j(\Phi_j,x))X_{ij} = C_i - \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\epsilon_j(\Phi_j,x) + \frac{\sum_{j=1}^n q_{ij}P_j\epsilon_j(\Phi_j,x)(1-\Delta_j))}{\theta_{i,min}}$$

Define $\eta_j(\Phi_j,x_{-i}) = \epsilon_j(\Phi_j,x) - \frac{x_{ij}q_{ij}}{\Phi_j}$. Equation (16) is equivalent to:

$\sum_{j=1}^n \alpha_{ij}(1-\epsilon_j(\Phi_j,x))X_{ij} - \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\frac{x_{ij}q_{ij}}{\Phi_j} + \frac{\sum_{j=1}^n q_{ij}P_j\frac{x_{ij}q_{ij}}{\Phi_j}(1-\Delta_j))}{\theta_{i,min}} = C_i - \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\eta_j(\Phi_j,x_{-i}) + \frac{\sum_{j=1}^n q_{ij}P_j\eta_j(\Phi_j,x_{-i})(1-\Delta_j))}{\theta_{i,min}}$

which is in turn equivalent to

$\sum_{j=1}^n \alpha_{ij}(1-\epsilon_j(\Phi_j,x))X_{ij} - \sum_{j=1}^n \frac{\alpha_{ij}q_{ij}}{\Phi_j}X_{ij} + \sum_{j=1}^n X_{ij}\frac{q_{ij}}{\Phi_j\theta_{i,min}} = C_i - \sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\eta_j(\Phi_j,x_{-i}) + \frac{\sum_{j=1}^n q_{ij}P_j\eta_j(\Phi_j,x_{-i})(1-\Delta_j))}{\theta_{i,min}}$

To get a more compact expression, let

$$\eta(\Phi,x_{-i}) = -\sum_{j=1}^n \alpha_{ij}q_{ij}P_j(1-\Delta_j)\eta_j(\Phi_j,x_{-i}) + \frac{\sum_{j=1}^n q_{ij}P_j\eta_j(\Phi_j,x_{-i})(1-\Delta_j))}{\theta_{i,min}}$$

It thus follows that (16) is equivalent to:

$$\sum_{j=1}^n \alpha_{ij}(1-\epsilon_j(\Phi_j,x) - \frac{\alpha_{ij}q_{ij}}{\Phi_j} + \frac{q_{ij}}{\Phi_j\theta_{i,min}})X_{ij} = C_i + \eta(\Phi,x_{-i})$$

Letting now $\delta_j(\Phi_j,x) = \epsilon_j(\Phi_j,x) - \frac{\alpha_{ij}q_{ij}}{\Phi_j} + \frac{q_{ij}}{\Phi_j\theta_{i,min}}$, equation (16) is finally equivalent to:

$$\sum_{j=1}^n \alpha_{ij}(1-\delta_j(\Phi_j,x))X_{ij} = C_i + \eta(\Phi,x_{-i}) \tag{17}$$

an expression, up to functions in $\frac{1}{\Phi_j}$, identical to equation (12) and note that $\delta_j(\Phi_j, x) \leq \delta_j(\Phi_j, 1)$ for all $x \in [0,1]^{np}$.

By assumption, $\alpha_{i1} > \alpha_{i2} > ... > \alpha_{in}$, i.e., for all $j \in [2,..,n]$, $\frac{\alpha_{ij}}{\alpha_{ij-1}} < 1$ so that there exists $\delta^0_{ij} > 0$ such that $\frac{\alpha_{ij}}{\alpha_{ij-1}} = 1 - \delta^0_{ij}$. We know that for all i and all j, $\lim_{\Phi_j = \infty} \delta_{ij}(\Phi_j, 1) = 0$. There thus exists $\Phi^{i,0}_j$ such that for all $\Phi_j \geq \Phi^{i,0}_j$, $\delta_{ij}(\Phi_j, 1) < \delta^0_{ij}$. As a result, for all $\Phi^i \geq \Phi^{i,0} = (\Phi^{i,0}_1, .., \Phi^{i,0}_n)$ and all $x \in [0,1]^{np}$, we have

$$\alpha_{i1}(1 - \delta_{i1}(\Phi^i_1, x)) > \alpha_{i2}(1 - \delta_{i2}(\Phi^i_2, x)) > ... > \alpha_{in}(1 - \delta_{in}(\Phi^i_n, x)) \tag{18}$$

Since bank i is seeking to minimize $\sum_{j=1}^n X_{ij}$ we are back in Proposition 1, that is, it is optimal to sell assets by decreasing risk weights. For this result to be true for all banks, it suffices to take $\Phi^0_j = \sup_i \Phi^{i,0}_j$, and $\Phi \geq \Phi^0 = (\Phi^0_1, .., \Phi^0_n)$. □

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