Economic foundations of generalized games with shared constraint: Do binding agreements lead to less Nash equilibria?

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Abstract

A generalized game is a situation in which interaction between agents occurs not only through their objective function but also through their strategy sets; the strategy set of each agent depends upon the decision of the other agents and is called the individual constraint. As opposed to generalized games with exogenous shared constraint literature pioneered by [Rosen, 1965], we take the individual constraints as the basic premises and derive the shared constraint generated from the individual ones, a set K. For a profile of strategies to be a Nash equilibrium of the game with individual constraints, it must lie in K. But if, given what the others do, each agent agrees to restrict her choice in K, something that we call an endogenous shared constraint, this mutual restraint may generate new Nash equilibria. We show that the set of Nash equilibria in endogenous shared constraint contains the set of Nash equilibria in individual constraints. In particular, when there is no Nash equilibrium in individual constraints, there may still exist a Nash equilibrium in endogenous shared constraint and we give two economic applications of this (elementary) result to collective action problems (carbon emission and public good problems).

Keywords: Generalized games, binding agreements, individual and shared constraints, collective action problems

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1 Introduction

It has long been recognized in Economics, Political sciences, International relations and more generally in Social sciences that Game theory is a useful tool to understand (and/or to predict) the outcome of a particular economic or social interaction between a set of agents (see [Fudenberg and Tirole, 1991], [Brams, 2011], [Schelling, 1980], [Moulin, 1986] for classical textbooks). A striking feature of most (not to say all) applied game theoretical models is that interaction between agents only takes place through the objective functions (utility or cost function). Given what the others choose, the aim of a given agent is to optimize her objective function by choosing the optimal strategy in a given set assumed to be invariant with respect to the choice of the other agents.

When each agent explicitly faces for instance a common binding constraint, the decision of a given agent may not only impact the objective function of the other agents but also their strategy set. Consider the well-known example of international environmental agreements (such as the Kyoto protocol) in which the total volume of emissions of greenhouse gas must be lower than a given threshold $\overline{e}$. From the point of view of a given country i, given the sum of emissions of greenhouse gas of the other countries e_{-i}, its total emission e_i explicitly depends upon the emissions of the other since the strategy set of country i is equal to $S_i(e_{-i}) = [0, \overline{e} - e_{-i}]$. Within such a simple and natural framework with a collective binding constraint, interaction between agents may not only take place through the objective function but also through their strategy set (see [Breton et al., 2006] and [Tidball and Zaccour, 2005] for early economic applications). Games in which the interaction takes place not only through the utility function (or cost function) but also through the strategy sets are called generalized games or Generalized Nash Equilibrium Problem (GNEP for short) (see [Facchinei and Kanzow, 2007], [Fischer et al., 2014]). Throughout this paper, we may call interchangeably GNEPs and generalized games.

Generalized games (GNEPs) are not recent and have received considerable attention in operational research. For instance, in their well-known survey, [Facchinei and Kanzow, 2007] offer a historical overview of GNEPs dating back to the seminal paper of [Arrow and Debreu, 1954] and [Nash, 1950] and [Nash, 1951] and they provide interesting examples of applications of such games in telecommunication or in environmental pollution. In the applied maths literature more generally, there has been an abundant number of articles on GNEPs in recent years either proposing new methods, existence results or numerical algorithms to find a Nash equilibrium (e.g., [Facchinei et al., 2009], [Aussel and Dutta, 2008], [Fischer et al., 2014]). However, to the best of our knowledge, there has been only few papers trying to apply GNEPs in Economics (but see [Breton et al., 2006], [Elfoutayeni et al., 2012], [Le Cadre et al., 2020]).

A generalized game with *individual constraints* can be naturally defined as a game in which each agent has to satisfy her own individual constraint, that is, each agent i is required to pick a strategy x_i in a set which is (possibly) dependent on the strategies picked by the other agents. In this sense, classical games but also GNEPs can be seen as generalized games with individual constraints.

A generalized game with *shared constraint* is a specific category of games with individual constraints and has been introduced for the first time in [Rosen, 1965] although the term shared constraint does not appear[1]. In his seminal paper, [Rosen, 1965] pioneered such games and proves an existence result about concave games. Remarkably, Rosen's result is true not only on E defined as the classical Cartesian product of strategy sets but also on any closed and bounded convex subset $X \subset E$. Later on in the literature, X has been called the *shared constraint set.* Such a game is called generalized game with shared constraint in the sense that all the agents share the same constraint, that is, the profile of strategies $\boldsymbol{x} := (x_1, ..., x_n)$ must always remain in the shared constraint set X: given x_{-i}, each agent i is required to pick a strategy $x_i \in E_i$ such that $\boldsymbol{x} := (x_i, x_{-i}) \in X$.

The striking feature of the shared constrained approach developed in [Rosen, 1965] and the subsequent literature is that this shared constraint set X is *exogenously given* and bears no relationship with any possible individual constraints. In [Rosen, 1965], the author never makes use of the notion of an individual constraint. Later on, initiated by [Bensoussan, 1974], [Harker, 1984] and [Harker, 1991] introduced a variational formulation of the equilibrium of a generalized game with shared constraint and the literature on GNEPs now formulate the equilibrium as a (quasi) variational inequality. In its most basic version (see [Fischer et al., 2014] or [Facchinei and Kanzow, 2007] for excellent review papers), the variational formulation involves the partial derivative of the objective function of each agent with respect to her own strategy, which means that the underlying functions (objective function/constraints function) have to be differentiable.

It is the aim of this (methodological) paper to show the fruitfulness of such generalized games in the Economics of binding constraints. We re-consider GNEPs with shared constraints and in particular we shed light on the relationships between the individual constraints on the one hand and the shared constraints on the other hand. Instead of considering a shared constraint set which is exogenously given and imposed to the set of agents, as in the literature on GNEPs, we consider these individual constraints as the basic premises of the game and derive an *endogenous shared constraint set* generated precisely by these individual constraints. The building of such a shared constraint set from basic individual constraints is the core of our article and main result, althoug elementary. From a pure economic point of view, the existing literature on generalized games with shared constraints has two limitations. First, as said, the shared constraint is in general postulated and not generated from the individual constraints[2]. Second, by formulating the equilibrium of the game as a variational inequality, the characterization of the equilibrium excludes the simplest games in which the strategy set of each agent is a finite set (e.g., the 2-2 games).

We prove in this paper the following interesting result for which the economic applications are numerous: the set of Nash equilibria of a generalized game with individual constraints is *included*

[1] Historically, generalized games were first introduced by [Debreu, 1952]. In [Harker, 1991] or in [Krawczyk, 2007], the authors note that a number of different names appeared in the literature to define these generalized games, abstract economy, social equilibria games, pseudo-Nash equilibria games or normalized equilibria ([Rosen, 1965]).

[2] For instance, in [Tidball and Zaccour, 2005], they consider a game with individual constraints, but, as we shall see, the shared constraint is not derived from the individual constraints.

in the set of Nash equilibria of the generalized game with shared constraint generated from these individual constraints. It happens that [Feinstein and Rudloff, 2021] proved independently a similar result (see Theorem 3.2. in their paper). This result, whose proof turns out to be simple, has two basic important consequences.

1. There are situations in which there is no Nash equilibrium in a game with individual constraints while such Nash equilibria exist in the game with shared constraint generated from the individual ones.

2. If there is no Nash equilibrium in shared constraint, then, no Nash equilibrium in the game with individual constraints can exist (the converse is however not true).

From an economic point of view, our result requires some binding agreements; given what the others do, each agent i agrees to pick a strategy not in her basic set of strategy (individual constraint) but in the shared constraint set that results from these individual constraints. Within our generalized game, sharing the constraint means in general that, given what the others do, compared with the primitive individual constraints X_i, each agent will now have to choose a strategy in a *smaller set*, that is, in $K_i \subset X_i$. Some strategies that were available in the game with individual (primitive) constraints are not anymore available in the game with shared constraint. Put it differently, introducing a shared constraint is equivalent to introduce restrictions and this kind of restriction exactly fits the notion of mutual restraint discussed in the well-known book of [Barrett, 2007] entitled *Why cooperate?*. Contrary to what the basic intuition could suggest, this form of mutual restraint typically generates more Nash equilibria and not less.

It is clear that this binding agreement requires some form of cooperation between the agents. In his well-known textbook, [Moulin, 1995] considers three modes of cooperation between a set of agents, direct agreements, decentralized behavior and justice. In this paper, while not explicitly modeled, the mode of cooperation considered is the first one, direct agreements, and can be thought (from a game theoretical point of view) of as the result of preplay communication[3]. These direct binding agreements are particularly important when there is no Nash equilibrium in the game with individual constraints while (at least) one equilibrium exists in the game with shared constraint. We illustrate this idea in the second part of the paper to collective action problems, an externality-pollution and a public good problem. In each model, we show situation in which there is no Nash equilibrium in individual constraints while there may be (at least) one Nash equilibrium in shared constraint.

The remainder of the paper is structured as follows. In Section 2, we remind the definitions of generalized games as well as the corresponding notions of Nash equilibria. We then define the notion of generalized game with shared constraint generated from a game with individual constraints,

[3] As observed in [Moulin, 1995], transaction cost is a drawback to this direct agreement mode. If this preplay communication is long and difficult, the transaction cost will be high. This problem of commitment is also discussed in the well-known book of [Ostrom, 1990]

something we call endogenous shared constraint and we establish the main finding of our paper, which states that the Nash equilibria of a generalized game with individual constraints are included in the set of Nash equilibria of the generalized game with endogenous shared constraint. In Section 3, we offer two different collective action problem models which illuminates our main result (simple to prove) when no Nash equilibrium exists in individual constraints. The first model is a public good problem while the second one is an environment control problem and we show the economic usefulness of the introduction of an endogenous shared constraint in these problems. Section 4 concludes the paper.

2 Games with individual and shared constraints

2.1 Generalized games with individual constraints

We consider a game with $N \geq 2$ agents (or players) and we denote $J = \{1, ..., N\}$ the set of players. The decision (or control) variable of each player $i \in J$ is denoted by $x_i \in E_i$, where E_i is a subset of $\mathbb{R}^{n_i}$ called the strategy set. Let $E = \prod_{i=1}^{N} E_i = E_1 \times \ldots \times E_N$ and denote by $\boldsymbol{x} \in E$ the vector formed by all these decision variables (strategies) which has dimension $n := \sum_{i=1}^{N} n_i$ so that $E \subset \mathbb{R}^n$. As usual in game theory, we denote by $x_{-i} \in E_{-i}$ the vector formed by all the players' decision variables except those of player i. To emphasize the i-the player's strategy, we sometimes write $(x_i, x_{-i}) \in E_i \times E_{-i}$ instead of $\boldsymbol{x} \in E$. Each player i has an objective function $\theta_i : \mathbb{R}^n \to \mathbb{R}$ that depends on both his own decision variables x_i as well as on the decision variables x_{-i} of all other players. We denote the objective function of player i by $\theta_i(x_i, x_{-i})$. For a given $x_{-i} \in E_{-i}$ and depending upon the game, the aim of agent i may be to maximize or to minimize its objective function $\theta_i(x_i, x_{-i})$. In general, when this objective function is a utility (or a payoff) function, it is the aim of the agent to maximize it while when it is a cost (or a loss) function, it is the aim of the agent to minimize it. Throughout the article, unless otherwise specified, we will assume that the objective function is a cost function so that each agent i, given the other players' strategies x_{-i}, is seeking a strategy x_i to minimize $\theta_i(x_i, x_{-i}) = \theta_i(\boldsymbol{x})$. Throughout the paper, we may use interchangeably the terms decision, decision variable and strategy and we only consider the case of pure strategy, that is, the situation in which each agent chooses a strategy with probability one.

In a generalized game, each player $i \in J$ must pick a strategy $x_i \in X_i(x_{-i}) \subseteq E_i$ where the set $X_i(x_{-i})$ explicitly depends upon the rival players' strategies. As in classical games in which the strategy set E_i of each agent i is given, in generalized games, the strategy set that we call the individual constraint function $X_i(x_{-i})$ (or simply the individual constraint) is also exogenously given. To define in full generality the individual constraint which depends upon the decision of others, let X_i be defined as follows:

$$X_i : E_{-i} \to \mathcal{P}(E_i) \quad i = 1, 2, ..., N \tag{1}$$

where $\mathcal{P}(E_i)$ is the power set of E_i. It is usual to call X_i a point-to-set map (ot a set-valued map) since it associates a subset of E_i to each point of E_{-i}. This means that for a given point $x_{-i} \in E_{-i}$, the strategy set also called the individual constraint of agent i is equal to $X_i(x_{-i}) \subseteq E_i$, a subset of E_i such as an interval. At a more abstract mathematical level, $X_i(x_{-i})$ may be a finite or countable union of intervals if we think of $X_i(x_{-i})$ as a Borel set. While interesting, this kind of mathematical generality is not our focus. We want instead to focus on the economic foundation of generalized games. At an economic level, in a strategic interaction with two agents, given the choice x_2 of agent 2, agent 1 may have to choose x_1 in E_1 subject to a constraint of the form $g_1(x_1, x_2) = x_1 + bx_2 \leq r_1$. Put it differently, the strategy set of agent 1 explicitly depends upon the choice of agent 2. Let us consider two economic examples of such a situation that will be considered in detail in section 3.

- In an environmental problem with externalities, agent 1 (i.e., country 1) may be constrained to choose its emission of greenhouse gas x_1 subject to a constraint of the form $g_1(x_1, x_2) = x_1 + b_2 x_2 \leq r_1$ where r_1 is the maximum emission of country 1 and $b_2 x_2$ is the impact emission of country 2 on country 1.

- In a public good problem, following [Guttman, 1978], each agent $i \in \{1, 2\}$ may have to choose a flat contribution x_1 plus a matching rate bx_2 (which is a function of the flat contribution of agent 2 where $b \in (0, 1]$) so that $g_1(x_1, x_2) = x_1 + bx_2 \leq r_1$ is the budget constraint of agent 1. A similar budget constraint holds for agent 2.

When thinking about collective action problems, there are various situations in which the decision of a given agent has an impact on the constraint of another agent.

Definition 1 *The 4-uplet $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$ is called a generalized game with individual constraints.*

It should be clear "classical games" encountered in Economics appear as a particular generalized games with individual constraints. When, for each $i \in J$, X_i is invariant with respect to the choice of the other agents, we are back to a classical game in which the interaction only takes place through the objective functions. In such a case, X_i reduces to E_i.

2.2 Generalized games with shared constraints: endogenous versus exogenous shared constraint

In a generalized game, it may be the case that for some x_{-i}, the strategy set of agent i is simply empty, that is, $X_i(x_{-i}) = \emptyset$, which means that the objective function is undefined so that the equilibrium can not exist. Such an empty set problem never occurs in classical games since the strategy set of each agent i is invariant with respect to x_{-i}. When $\boldsymbol{x} \in E$ is such that $x_i \in X_i(x_{-i})$ for each agent $i \in J$, we say that the profile of strategies $\boldsymbol{x}$ is *admissible*, which means that it is a candidate point to be an equilibrium of the generalized game.

Definition 2 *For a given generalized game with individual constraints $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$, let K be the subset of E defined as follows.*

$$K = \{\boldsymbol{x} \in E,\ \forall i \in J,\ x_i \in X_i(x_{-i})\} \tag{2}$$

K is called the set of admissible strategies of the generalized game with individual constraints.

The set K represents the set of profiles of strategies $\boldsymbol{x}$ for which the generalized game with individual constraints is defined for all agents. If $\boldsymbol{x}$ does not belong to K, it can not be a Nash equilibrium of the generalized game. Throughout the paper, for the sake of interest, we assume that K is not empty.

As in classical games, each agent i, given the other players' strategies x_{-i}, is seeking a strategy x_i to optimize $\theta_i(x_i, x_{-i})$ subject to the constraint $x_i \in X_i(x_{-i})$. A generalized Nash equilibrium problem (GNEP) is therefore the given of N constrained optimization problems, that is, for each $i \in J$, given x_{-i}, agent i optimizes $\theta_i(x_i, x_{-i})$ subject to $x_i \in X_i(x_{-i})$. A Nash equilibrium $\boldsymbol{x}^* = (x_1^*, ..., x_N^*) \in K$ of the generalized game thus is such that no agent i wants to unilaterally deviate from her part of the equilibrium profile $\boldsymbol{x}^*$ but also such that the constraint of each agent $i \in J$ is satisfied, i.e., $\boldsymbol{x}^* \in K$. The following definition makes clear this constraint.

Definition 3 *The profile of strategies $\boldsymbol{x}^* \in E$ is a Nash equilibrium of the generalized game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ if, for each $i \in J$ and each $x_i \in E_i$ such that $x_i \in X_i(x^*_{-i})$, it holds true that $\theta_i(x_i^*, x^*_{-i}) \leq \theta_i(x_i, x^*_{-i})$.*

From the above discussion, a necessary but not sufficient condition on the profile of strategies $\boldsymbol{x}$ to be a Nash equilibrium is that $\boldsymbol{x} \in K$. Since K is assumed to be not empty, it makes economic sense to require from each agent that given what the other agents are choosing, i.e., x_{-i}, agent i should pick a strategy x_i such that the profile $\boldsymbol{x} = (x_i, x_{-i})$ lies in K. Given x_{-i}, let $K_i(x_{-i})$ be the set of strategies of agent i defined as follows

$$K_i(x_{-i}) = \{x_i \in X_i(x_{-i}) : \boldsymbol{x} \in K\} \tag{3}$$

We are now in a position to define a generalized game with endogenous shared constraint, that is, a generalized game in which the shared constraint is generated from the individual constraints.

Definition 4 *The 4-uplet $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ is called a generalized game with shared constraint generated from the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$. We call such a game a game with endogenous shared constraint.*

We are now in a position to give the definition of a Nash equilibrium in a game with endogenous shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$. The following definition is similar to the one given for a generalized with individual constraints except that each agent i is required to choose a strategy in $K_i(x_{-i})$ rather than in $X_i(x_{-i})$.

Definition 5 *The profile of strategies $\boldsymbol{x}^* \in E$ is a Nash equilibrium for the generalized game with endogenous shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ if for each $i \in J$ and each $x_i \in E_i$ such that $x_i \in K_i(x^*_{-i})$, we have $\theta_i(x_i^*, x^*_{-i}) \leq \theta_i(x_i, x^*_{-i})$.*

To see the difference between a game with individual constraints and a game with endogenous shared constraint, assume that $X_i(x_{-i}) := \{x_i \in E_i : g_i(\boldsymbol{x}) \leq 0\}$ is the strategy set (or constraint) of each agent i. The admissible set K (see equation (2)) can be rewritten as $K = \{\boldsymbol{x} \in E, \ \forall i \in J, g_i(\boldsymbol{x}) \leq 0\}$ so that $K_i(x_{-i}) := \{x_i \in E_i : \forall j \in J, g_j(\boldsymbol{x}) \leq 0\}$.

- In the game with individual constraint, given x_{-i}, each agent optimizes its objective function with respect to x_i subject to its *own individual constraint* $g_i(\boldsymbol{x}) \leq 0$, that is, $x_i \in X_i(x_{-i})$.
- In the game with shared constraint generated from the individual ones, given x_{-i}, each agent optimizes its objective function with respect to x_i subject to $g_1(\boldsymbol{x}) \leq 0$, $g_2(\boldsymbol{x}) \leq 0$,..., $g_N(\boldsymbol{x}) \leq 0$, that is, $x_i \in K_i(x_{-i})$. Given x_{-i}, when each agent i chooses a strategy x_i, she takes not only into account its own constraint but also the *constraints of all the other agents* so that (x_i, x_{-i}) lies in K.

It should be clear in a game with endogenous shared constraint, the set of strategies of each agent i may be *reduced* compared to the game with individual constraints, that is,

$$K_i(x_{-i}) \subseteq X_i(x_{-i}) \quad \forall i \in J \tag{4}$$

In general, the inclusion may be strict for some agents, that is, as long as $X_i(x_{-i})$ is not empty, $K_i(x_{-i}) \subset X_i(x_{-i})$. In what follows, to emphasize that the set K is the shared constraint, we may denote the game $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$ as $(J, E, (\theta_i)_{i\in J}, K)$.

Remark 1 *Within our approach, we take the set $X_i(x_{-i})$ (the individual constraints) as the basic premises and we derive the set K (the endogenous shared constraint) from these individual constraints. This is in sharp contrast with the literature on generalized games in which the shared constraint is exogenous. Following [Rosen, 1965], authors typically start with a classical game for which $E = \prod_{i=1}^{N} E_i$ and what they call the shared constraint is an exogenous (mathematically convenient) prescribed set $X \subset E$, that is, the profile of strategies $\boldsymbol{x}$ must be located in X, see e.g., [Fischer et al., 2014] for nice review papers.*

In the literature on generalized games, the *exogenous* shared constraint X (i.e., such that $X \subset E$) thus appears as the basic premise. Given this exogenous shared constraint X, knowing x_{-i}, each agent i thus is required to choose a strategy x_i in X, and this means that her strategy set now is defined as $X_i(x_{-i}) = \{x_i \in E_i : (x_i, x_{-i}) \in X\}$. As opposed to the approach followed in this paper, the individual constraints $X_i(x_{-i})$ are derived from the exogenous shared constraint. Let $(J, E, (\theta_i)_{i\in J}, X)$ be a generalized game with an exogenous shared constraint. The following well-known result about the existence of Nash equilibria for concave n-person games is due to [Rosen, 1965]. Using our terminology, the following result is an existence result for a game with an exogenous shared constraint X.

Theorem 1 *([Rosen, 1965]) Let $(J, E, (\theta_i)_{i\in J}, X)$ be a game with an exogenous shared constraint where θ_i is a payoff function. If the set X is convex, closed and bounded and if each player's payoff* *function $\theta_i(x_i, x_{-i})$, $i \in J$ is continuous and concave in x_i, then, the generalized game has at least one Nash equilibrium.*

[Rosen, 1965] considers the simple example of a 2-person game in which X is a compact convex subset of the unit square such as an ellipse. This thus means that given the choice of agent 1, agent 2 has to choose a number between zero and one such that the couple of numbers chosen must be located in the ellipse. In many other papers, the authors consider the X defined as $X = \{x \in E \mid G(x) \leq 0\}$, where $G : \mathbb{R}^n \to \mathbb{R}$ is a component-wise convex function called the shared constraint function, an assumption convenient for the mathematical analysis of the generalized game (see [Facchinei and Kanzow, 2010] or [Fischer et al., 2014] for review papers). An attractive feature of these games with shared constraint is that they are analytically convenient. Under particular assumptions on X and on the objective functions θ_i, $i \in J$, it is often possible to use a fixed-point theorem (Kakutani, Tarski...) to prove the existence of Nash equilibrium.

2.3 Shared constraint and binding agreements

As discussed in the introduction of this paper, sharing the constraints requires a particular form of cooperation between agents called *direct agreements* in [Moulin, 1995]. Within our generalized game framework, given what the others do, each agent must agree to restrict their choice in the shared constraint K, that is, given x_{-i}, each agent i must pick a strategy x_i in $K_i(x_{-i})$ rather than in $X_i(x_{-i})$ where $K_i(x_{-i})$ is typically included in $X_i(x_{-i})$. Given x_{-i}, it might be the case that the best response of a given agent i lies in $X_i(x_{-i})$ and not in $K_i(x_{-i})$. As a result, this self-restriction to the set of strategies $K_i(x_{-i})$ rather than $X_i(x_{-i})$ requires some binding agreements that are however not explicitly modeled here. These binding agreement problem is identical to the problem of credible commitment discussed in the well-known book of [Ostrom, 1990] in which she notes that one (frequent theoretical) solution to this problem is *coercion*. Roughly speaking, one can make a (perhaps disputable) distinction between two types of coercion, external or internal.

- external (or exogenous) coercion corresponds to the situation in which each agent must comply with law or regulation and this means that there is an external enforcer to use the terminology of ([Ostrom, 1990]). In such a situation, an agent who does not comply with the law or regulation can be sanctioned (i.e., fined) by the external enforcer.

- Internal (or endogenous) coercion corresponds to the situation in which a group of agents (employees, firms, countries or even the overall society itself) must reach an agreement without any external enforcer. This clearly means that such an agreement is based on voluntarism since an agent who breach the (contractual) agreement can not be fined.

Internal coercion thus is an agreement based on self-restriction (i.e., something that must (or must not) be done) which turns out to be very similar to the notion of *mutual restraint* discussed in [Barrett, 2007] (see chapter five). In [Barrett, 2007], the author explicitly considers the case in which agents are countries that seek to supply global public goods such as nuclear non-proliferation or climate change mitigation (e.g., limit carbon emission). For instance, when one considers a set of countries that try to reduce their individual pollution, mutual restraint can be reached through international treaties and these treaties can be thought of as an example of an internal coercion[4]. Given x_{-i}, the restriction of each agent i to the set $K_i(x_{-i})$ can be seen as a possible formalization of the notion of mutual restraint discussed in [Barrett, 2007].

In what follows, we thus make the implicit assumption that agents succeeded to reach binding agreements so each agent (on a voluntary basis given x_{-i}) agrees to be restricted to $K_i(x_{-i})$. As we shall see, when no equilibrium exists in the game with individual constraints, this mutual restraint might be the unique solution to reach an equilibrium situation. In the last section of this paper, we shall offer two different models of collective actions in which the equilibrium only exists (depending upon parameters) in shared constraint.

2.4 An elementary game theoretical result

As we shall now see, a striking feature of a game with endogenous shared constraint, compared with the game with individual constraints, is that it may possess *additional Nash equilibria*. We already know that for a profile $\boldsymbol{x}$ to be a Nash equilibrium, $\boldsymbol{x}$ must be in K. But if each agent i, given x_{-i}, agrees to choose x_i such that $\boldsymbol{x} = (x_i, x_{-i})$ is in K (i.e., $x_i \in K_i(x_{-i})$), contrary to the intuition, the set of Nash equilibria may be *larger*. This is the basic statement of the following result

Proposition 1 *The set of Nash equilibria of a game with individual constraints is included in the set of the Nash equilibria of the game with shared constraint generated from the individual constraints, that is, if $\boldsymbol{x}^* = (x_1^*, ..., x_N^*) \in K$ is a Nash equilibrium for the game with individual constraints, it is also a Nash equilibrium for its generated game with shared constraint but the converse is not true.*

Proof. See the appendix.

Independently, [Feinstein and Rudloff, 2021] proved a similar result (see Theorem 3.2. in their paper). To see that the converse is not true, consider the following example of a game in an endogenous shared constraint. Let $J = \{1, 2\}$ and let $E_1 = E_2 = [0, 1]$ so that $E = [0, 1] \times [0, 1]$ is basic the strategy space. Assume as before that the objective function of each player is a cost function to be minimized. The cost functions are respectively $\theta_1(x_1, x_2) = x_1 + x_2$ and $\theta_2(x_1, x_2) = x_2 - x_1$ for agent 1 and 2. Assume now that the individual constraint for agent 1 is $g_1(x_1, x_2) = x_1 + x_2 \leq 1$ while it is equal to $g_2(x_1, x_2) = x_1 + x_2 \geq \frac{1}{2}$ for agent 2. In this example,

[4]In this chapter five devoted to mutual restraint, [Barrett, 2007] focuses on the prevention on the possible use of nuclear weapons (e.g., by an unstable government of some country) and notes that in order "to prevent nuclear weapons from spreading, the security of non-nuclear states must somehow be assured". The way to implement such a nuclear non-proliferation commitment must be done through binding agreements, that is, through a *treaty* such as the North Atlantic Treaty Organization (NATO). Interestingly, All NATO decisions are made by consensus (after discussion and consultation among member countries) and a "decision reached by consensus is an agreement reached by common consent". See https://www.nato.int/nato-welcome and note that there are currently 30 member states.

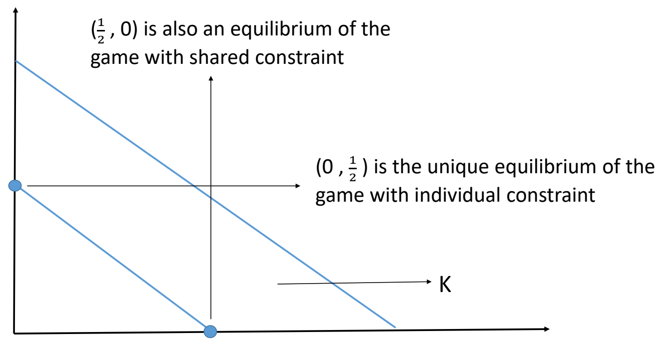

Figure 1: One equilibrium in individual constraints, two equilibria in shared constraint

the set K is defined as

$$K = \left\{ (x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1 \ \text{ and } \ x_1 + x_2 \geq \frac{1}{2} \right\} \tag{5}$$

It is not difficult to see that K is a compact and convex set (see Fig. 1) and that the profile of strategies $(0, \frac{1}{2}) \in K$ is a Nash equilibrium of the game with individual constraints. Consider now the profile of strategies $(\frac{1}{2}, 0) \in K$.

- In the game with individual constraints, $(\frac{1}{2}, 0) \in K$ is *not* a Nash equilibrium. To see this, it suffices to note that when $x_2 = 0$, the best response of agent 1 is 0 and the constraint of agent 1 is fulfilled since $g_1(x_1, x_2) = x_1 + x_2 \leq 1$. Since $\frac{1}{2}$ is not the best response, $(\frac{1}{2}, 0) \in K$ thus is not an Nash equilibrium.

- In the game with shared constraint, $(\frac{1}{2}, 0) \in K$ is a Nash equilibrium. To see this, note that if $x_1 = \frac{1}{2}$, then, the best response of agent 2 is 0. This choice of 0 minimizes the cost of agent 2 and satisfies the constraint of agent 2 since $\frac{1}{2} + 0 \geq \frac{1}{2}$ but also the constraint of agent 1 since $\frac{1}{2} + 0 \leq 1$. If $x_2 = 0$, the best response of agent 1 is now to choose $\frac{1}{2}$ because agent 1 takes into account the constraint of agent 2, i.e., $g_2(x_1, x_2) = x_1 + x_2 \geq \frac{1}{2}$. As opposed to the game with individual constraints, agent 1 can not choose 0. As a result, $(\frac{1}{2}, 0)$ is an Nash equilibrium for the game with endogenous shared constraint.

Contrary to the basic intuition one may have, proposition 1 says that adding more constraints in a game, that is, requiring from each agent i to choose a strategy in $K_i(x_{-i}) \subset X_i(x_{-i})$, may actually *expand* the set of Nash equilibria and not the opposite. Proposition 1 thus yields the two following basic insights.

1. There may be situations in which there is no Nash equilibrium in the game with individual constraints while there exists a Nash equilibrium in the game with endogenous shared constraint (generated from the individual ones).

2. If no Nash equilibrium exists in the game with endogenous shared constraint, no Nash equilibrium exists in the game with individual constraints (the converse is however not true).

While proposition 1 is elementary to prove from a mathematical point of view, it is interesting to note that it does not require the underlying functions to be differentiable, as opposed to the variational formulation of the Nash equilibrium in generalized games[5]. Proposition 1 thus can also be applied to generalized games in which the set of strategies of each agent is finite, for which the objective function needs not be a continuous function. In classical game theory–games for which the set of strategies E can be written as the "full" Cartesian product of individual strategy set E_i–there is a fairly important body of literature[6], initiated by [Dasgupta and Maskin, 1986], that prove the existence of a Nash equilibrium (in pure strategy) in discontinuous games. In this paper, we take an alternative road map and we submit the idea that when a Nash equilibrium does not exist in a game with individual constraints, its existence can be obtained simply by sharing the constraints rather than weakening the assumptions on the objective functions. Let us provide such an example.

Let $J = \{1, 2, ..., N\}$ be the set of agents. For each $i \in J$, the strategy set of agent i is $E_i = [0, 1]$ so that $E = [0, 1]^N$. Assume moreover that the characteristics of the agents are as follows.

1. Each agent $i \in J$ has a cost function (to be minimized) equal to $\theta_i(x_i) = x_i$.
2. Each agent $i \in J$ has the following constraint function.
   - If $x_j \geq 0.9$ for every $j \in J \setminus \{i\}$, then, $X_i(x_{-i}) = [\frac{1}{2}, 1]$.
   - If $x_j < 0.9$ for at least one $j \in J \setminus \{i\}$, then, $X_i(x_{-i}) = \emptyset$.

This game thus differs from classical ones encountered in economic theory in that the interaction only occurs through the set of strategies but not through the objective functions. From the specification of the game, if a given agent i, with $i \neq j$ chooses a number $x_i \geq 0.9$, the cost of agent i is simply equal to the number x_i chosen. If there is one agent j who chooses a number $x_j < 0.9$, with $i \neq j$ then, the set of strategies of a given agent is empty and the objective function thus is undefined[7]. Before discussing the outcome of the game with individual constraints, let us consider the game with shared constraint generated from the individual constraints. In this game with shared constraint, the set of strategies of each agent $i \in J$ is equal to $K_i(x_{-i}) = [0.9, 1]$ (see equation 3) so that K is not empty and equal to

$$K = \prod_i K_i(x_{-i}) = [0.9, 1]^N$$

[5]It should also be pointed out that the solutions of a variational inequality do not necessarily coincide with the Nash equilibrium of the generalized game, that is, a Nash equilibrium needs not be the solution of a variational inequality, see e.g., [Fischer et al., 2014], theorem 2.2 p. 525.

[6]We refer the reader to the introductory paper by [Reny, 2016] in the 2016-symposium of *Economic Theory*

[7]One can think of this abstract game theoretic framework to represent a tax competition problem in a fiscal union (see for instance [Zodrow, 2003]). State members may be committed to choose a tax rate greater than a given threshold. However, if one member state breaches the commitment, the problem becomes undefined for the other member states in that their strategy set is empty.

Since agents minimize a cost function, the profile of strategies $\boldsymbol{x}^* = (0.9, ..., 0.9)$ thus is the *unique* Nash equilibrium of the game with shared constraint $(J, E, (\theta_i)_{i\in J}, (K_i)_{i\in J})$. Therefore, from proposition 1, if the game with individual constraints $(J, E, (\theta_i)_{i\in J}, (X_i)_{i\in J})$ has a Nash equilibrium, it must necessarily be $\boldsymbol{x}^* = (0.9, ..., 0.9)$. But $\boldsymbol{x}^* = (0.9, ..., 0.9)$ is *not* a Nash equilibrium of the game with individual constraints. If $x_{-i} = (0.9, ..., 0.9)$, the best response of agent i is 0.5 and not 0.9, which means that $\boldsymbol{x}^* = (0.9, ..., 0.9)$ is not a Nash equilibrium of the game with individual constraints. When one agents picks a number lower than 0.9, the objective functions are undefined for the other agents and this means that there is no Nash equilibrium in such a game with individual constraints.

2.5 A Classification of games

In table 1, we offer a fruitful classification of games, encountered in Economics and Operations Research literature, through the way interaction is introduced.

In a generalized game, (see table 1), interaction occurs not only through the objective function but also through the strategy sets, that is, the objective function of a given agent $i \in J$ depends upon the decisions of the other players, i.e., $\theta_i(x_i, x_{-i})$ but the set of strategy of each agent i also depends upon the decisions of the other agents, that is, each agent i must choose $x_i \in X_i(x_{-i})$ where $X_i(x_{-i})$ is a subset of E_i. While not in the table, one can also distinguish generalized games with an exogenous shared constraint from generalized games with endogenous shared constraint, the subject of this paper.

In what we call a *semi-generalized game*, (see table 1), interaction may occur through the strategy sets or through the objective function but not both. When interaction occurs through the objective function but not through the strategy sets, we naturally call such a situation a *classical game* since it is the typical one encountered in most economic textbooks and papers in economic theory (see e.g. classic textbooks [Fudenberg and Tirole, 1991], [Moulin, 1986], [Myerson, 2013], [Osborne and Rubinstein, 1994]). When interaction occurs through the strategy set but not through the objective function, we call such a situation a *non-classical game* since, although fairly natural when one thinks to collective action problems, their application in the Economic literature is still limited. In the future, these kinf of generalized games could be widely applied. To the best of our knowledge, this type of game appeared for the first time in [Braouezec and Wagalath, 2019] in a (stress-test) regulatory framework although the authors do not mention that the underlying game they consider is an example of a generalized game. Finally, when there is no interaction at all, that is, when the objective function only depends upon the decision variable of agent i, that is $\theta_i(x_i)$ and when E_i is exogenously given (e.g., it is a compact set), this gives rise a non-strategic decision problem, the field of decision-theory (as opposed to game theory).

Types of interaction in games

Objective function \ Strategy set	Yes	No
Yes	Generalized game	Semi-generalized game
No	Semi-generalized game	Non-strategic decision problem

Table 1: Dependence through the objective function and/or through the strategy sets

3 Applications to collective action problems

We shall now discuss two interesting applications of games with endogenous shared constraint, one applied to an environmental problem and another applied to a public good problem. A striking feature of these two economic applications is that a Nash equilibrium may not exist in individual constraints while it always exists in endogenous shared constraint. The first example is applied to an environmental problem (it is indeed formulated as a non-classical game) and the general existence result follows from the fundamental theorem of [Rosen, 1965] (see theorem 1 in this paper). In the second example, applied to the financing problem of a public good, we consider a generalized game and show that when the dispersion of the optimal contribution of each agent is too large, no Nash equilibrium exists in individual constraints while a Nash equilibrium (indeed many) exists in endogenous shared constraint. From a mathematical point of view, to ease the analysis, we shall focus on the case in which the individual constraints are uni-dimensional.

3.1 Limiting global warming

Given the critical nature of the subject, the limitation of global warming of the earth, there is a large (game theoretical based) literature on the subject (see e.g., [Hoel and Schneider, 1997], [Carraro and Siniscalco, 1993], [Barrett, 2001]). We refer the reader to [Missfeldt, 1999], which is a survey of game theoretic models of trans-boundary pollution but note that this review paper do not mention generalized games. We found only few papers on the subject, [Tidball and Zaccour, 2005] and [Krawczyk, 2005] that analyze the pollution problem as a generalized game. In these models, the choice variable of a given country x_i is typically its pollution level (i.e., measured by the volume of emission of greenhouse gas) which, in the simplest case, is modeled as a linear function of its production. Such an environmental problem is interesting but challenging because the production of a given country typically generates *negative externalities* (i.e., pollution) to all the other countries.

In [Tidball and Zaccour, 2005], as seen earlier, they consider a model where each country seeks to maximize a profit function of the type $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + ... + x_n)$, where x_i represent the emissions of country i and are assumed to be proportional to the production of country i, $f_i(x_i)$ is a non-negative, twice-differentiable, concave and increasing function and the damage cost due to all the countries is denoted by a convex twice-differentiable increasing cost function $d_i(x_1 + ... + x_n)$. They consider three types of problems which correspond to three types of constraint:

- A *Generalized Nash equilibrium problem with individual constraints* where each agent is seeking to maximize $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + ... + x_n)$ subject to the constraint $x_i \leq E_i$ with E_i an exogenous given upper bound on emissions.

- A *cooperative scenario*, where agents agree to jointly maximize the sum of their profit function: $\max_{x_1,...,x_n} \sum_{i=1}^n f_i(x_i) - d_i(x_1 + x_2 + ... + x_n)$ subject to $\sum_{i=1}^n x_i \leq \sum_{i=1}^n E_i$.

- A *Generalized Nash equilibrium problem with an exogenous shared constraint* where each agent is seeking to maximize $w_i(x_1, ..., x_n) = f_i(x_i) - d_i(x_1 + x_2 + ... + x_n)$ subject to the constraint $\sum_{i=1}^n x_i \leq \sum_{i=1}^n E_i$.

The purpose of [Tidball and Zaccour, 2005] is to characterize and compare the solutions of these three different scenarios. They show that the Nash equilibrium in individual constraints may be better than the Nash equilibrium with (exogenous) shared constraint. In their framework, the shared constraint is actually not generated from the individual constraints[8]

In the same vein, [Krawczyk, 2005] proposes another model where three players $j = 1, 2, 3$ located along a river are engaged in an economic activity at a chosen level x_j and their joint production externalities must satisfy environmental constraints set by a local authority. It is assumed that player$j$ has a level of pollution $e_j x_j$, where e_j is the emission coefficient of player j. The pollution is expelled into the river and reaches a monitoring station in the amount of $\sum_{j=1}^3 \delta_{jl} e_j x_j$ where δ_{jl} is the decay-and-transportation coefficient from player j to location l, and it is assumed that there are two monitoring stations, $l = 1, 2$, and the local authority has set maximum pollutant concentration levels K_l. It gives the following Generalized Nash equilibrium problem with shared constraint: each player j is supposed to maximize its net profit $\phi_j(\boldsymbol{x}) = d_1 - d_2(x_1 + x_2 + x_3) - (c_{1j} + c_{2j} x_j) x_j$ (where $d_1, d_2, c_{1j} c_{2j}$ are given constants) under the shared constraint $\sum_{j=1}^3 \delta_{jl} e_j x_j \leq K_l$, $l = 1, 2$. [Krawczyk, 2005] proves existence of a unique Nash equilibrium under reasonable assumptions and exhibits an algorithms that converges towards the solution.

Inspired by [Tidball and Zaccour, 2005] and [Krawczyk, 2005], we now introduce a new environmental model formulated in terms of generalized with individual constraints and with endegenous shared constraint.

Consider the following model with $N \geq 2$ countries/economies. Let $x_i \in \mathbb{R}_+$ be the quantity of non-renewable energy (i.e., fossil energy) chosen by country i where the carbon emissions C_i by country $i \in \{1, 2, ..., N\}$ are proportional to x_i (similar to [Tidball and Zaccour, 2005] and [Krawczyk, 2005]), that is $C_i = c_i \times x_i$, where $c_i > 0$ (and note that $E = \mathbb{R}_+^N$). Let $f_i(x_i)$ be the profit function of country i as a function of x_i where f_i is an increasing concave function. On this

[8] The individual constraint of country i does not depend upon the choices of the other countries, for each $i \in J$, $X_i(x_{-i}) = [0, E_i]$ no matter what x_{-i} is. It thus follows that $K = \prod_{i \in J} X_i$ while the shared constraint is given by the set $S := \{\boldsymbol{x} \in K : \sum_{i \in J} x_i \leq \sum_{i \in J} E_i\}$. The game with shared constraint proposed by [Tidball and Zaccour, 2005] thus is *not* generated from the game with individual constraints since $\boldsymbol{x}$ can be in S but not in K. To see this within a numerical example, assume that $N = 2$ and that $\overline{x}_1 = 3$ and $\overline{x}_2 = 2$ so that $X_1 = [0, 3]$ while $X_2 = [0, 2]$. The point $(2, 3) \notin X_1 \times X_2$ while $(2, 3) \in S$.

aspect, we differ from [Tidball and Zaccour, 2005] and [Krawczyk, 2005] since we assume that the payoff function of a country only depends on x_i, the quantity of non-renewable energy chosen by i while the constraint depends upon $\boldsymbol{x} = (x_1, x_2, ..., x_N)$. More specifically, we assume that each country i has to satisfy an individual linear constraint of the form:

$$g_i(\boldsymbol{x}) := a_{ii} \times c_i \times x_i + \sum_{j \neq i} a_{ji} \times c_j \times x_j \leq S_i \tag{6}$$

where a_{ji} is a coefficient measuring how the use of fossil fuels by country j is impacting the environment of country i and S_i is a threshold determined for each country i. The constraint S_i may come from a regulatory institution. Note that in case in which $j = i$, the coefficient a_{ii} measures how the use of fossil fuels by country i is impacting its own environment. Given x_{-i}, the aim of country i is to maximize $f_i(x_i)$ subject to $g_i(x_i, x_{-i}) \leq S_i$ and note that, using our classification of games, such a strategic interaction is an example of a non-classical game.

The aim is now to study the possible existence of a Nash equilibrium for this game with individual constraints and its generated game with (endogenous) shared constraint. As seen earlier, it may indeed be the case that there is no Nash equilibrium for the game with individual constraints. To see this, assume that $N = 2$. Consider country 1 and assume that $a_{11} = c_1 = 1$ and that $a_{21} = 0$ so that its constraint is given by $x_1 \leq S_1$. Consider now country 2 and assume that $a_{22} = c_2 = 1$ and assume now that $a_{12} = 1$. From equation (6), it thus follows that the constraint of country 2 is given by $x_2 + x_1 \leq S_2$. Assume now that $S_1 > S_2$ and that this generalized game with individual constraints has at least a Nash equilibrium (x_1^*, x_2^*). Such an equilibrium must satisfy $x_1^* = S_1$. But in such a case, no matter what $x_2 \geq 0$ is, the constraint of country 2 is never satisfied which means that there is no Nash equilibrium in individual constraints.

We shall now derive a more general result about the (non) existence of a Nash equilibrium in this environmental game theoretical model with individual constraints. From the individual constraint given by equation (6), for a profile of strategy $\boldsymbol{x} \in \mathbb{R}_+^N$ to be Nash equilibrium, the constraints must be satisfied, that is, $\boldsymbol{x} \in \mathbb{R}_+^N$ must be such that $g_i(\boldsymbol{x}) \leq S_i$ for $i = 1, 2, ..., N$. From the monotonicity of the profit function of each country i, the constraint of each country will be binding at equilibrium $x^* = (x_1^*, ..., x_N^*)$, which means that for $i = 1, 2, ..., N$, $g_i(\boldsymbol{x}^*) = S_i$. The problem reduces to the analysis of a linear system of the form $A^T X_c = \boldsymbol{S}$ where A is the matrix of the coefficient $(a_{ij})_{i,j=1,2,...,N}$ (T indicates the transpose), $X_c = (c_1 x_1, .., c_N x_N) \in \mathbb{R}_+^N$ and $\boldsymbol{S}$ is the vector formed by S_i, $i = 1, 2, ..., N$. If $\boldsymbol{x}^* = (x_1^*, ..., x_N^*)$ is a Nash equilibrium for this game with individual constraints, it must satisfy the linear system $A^T X_c = \boldsymbol{S}$, that is:

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1N} \\ a_{22} & a_{22} & ... & a_{2N} \\ ... & ... & ... & ... \\ a_{N1} & a_{N2} & ... & a_{NN} \end{pmatrix}^T \begin{pmatrix} c_1 x_1^* \\ c_2 x_2^* \\ ... \\ c_N x_N^* \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix} \tag{7}$$

where $\boldsymbol{x}^* \in \mathbb{R}_+^N$.

The next result exhibits conditions under which there is no Nash equilibrium in individual constraints. In particular, when the above linear system has no solution in $\mathbb{R}_+^N$, then, no Nash equilibrium of the game with individual constraints can exist. However, in such a case in which the linear system has no solution, a Nash equilibrium of the game with endogenous shared constraint still exists.

Proposition 2 *The game with individual constraints has (at least) one Nash equilibrium x^* if and only if the linear system as defined in equation (7) admits at least one solution $X_c^* \in \mathbb{R}_+^N$. If the linear system given by equation (7) admits no solution in $\mathbb{R}_+^N$, then, the game with individual constraints has no Nash equilibrium.*

Proof. See the appendix.

In appendix, we give a complete characterization of the existence or not of Nash equilibria for the game with individual constraints. It is interesting at this stage to compare our framework with the one offered in [Tidball and Zaccour, 2005] and [Krawczyk, 2005]. Within our approach, as opposed to the two mentioned papers, a Nash equilibrium does not always exist in individual constraints. However, as we shall see, a Nash equilibrium in shared constraint always exists. In [Tidball and Zaccour, 2005], they show that Nash equilibria in individual constraints may be better in some sense than the ones in shared constraint but, as already discussed, their shared constraint is exogenous since it is not generated from the individual ones.

Let us now consider the game with shared constraint generated from our game with individual constraints in which each country k is seeking to maximize $f_k(x_k)$ subject to the endogenous shared constraint defined as $g_i(\boldsymbol{x}) := a_{ii} \times c_i \times x_i + \sum_{j \neq i} a_{ji} \times c_j \times x_j \leq S_i$, $i = 1, ..., N$. From proposition 1, we know that the Nash equilibria of the game with shared constraint contains the Nash equilibria of the game with individual constraints. But Nash equlibria of the game with individual constraints may not always exist, as seen in proposition 2. In the next result, thanks to the existence result by Rosen for n-person concave games, we show that there always exists Nash equilibria for our game with endogenous shared constraint.

Proposition 3 *The game with a shared constraint generated from the individual ones always has at least one Nash equilibrium. In particular, a Nash equilibrium in the game with endogenous shared constraint exists whether the linear system given by equation (7) admits a solution or not.*

Proof. See the appendix.

Proposition 3 is an interesting result from a binding agreements point of view since it clearly shows that a Nash equilibrium always exists in the game with shared constraint generated from the individual constraints while it may fail to exist in the game with individual constraints. By reinforcing the environmental constraints, i.e., by sharing the constraints, each country takes into account the individual constraints of the other countries, and this binding agreement always generates at least one Nash equilibrium. As already discussed, this interesting feature is counter intuitive at first glance: by putting more constraints on each country (by considering an endogenous shared constraint rather than individual constraints), one may generate a larger set of Nash Equilibria.

3.2 Contributing to a public good

We now consider a model of collective action with $N \geq 2$ agents, similar to [Guttman, 1978] and to [Cornes and Hartley, 2007] (see also [Buchholz et al., 2011]) but in which each agent faces an individual constraint (see [Sandler, 2015] for a review paper on collective action models). Each agent i is asked to make a voluntary contribution to finance a public investment project such as a bridge, a street lighting or any public infrastructure equipment (water, gas, internet...) subject to an individual constraint. For concreteness, we assume that the higher the total contributions, the higher quality of the underlying investment. Following the standard terminology introduced in [Guttman, 1978] (see also ([Buchholz et al., 2011])), let $x_i \in E_i := \mathbb{R}^+$ be the flat (or direct) contribution of agent i and let $b \in [0,1]$ the known percentage of the sum of the flat contributions of all the other agents, which thus defines here indirect contribution. The indirect contribution of a given agent i, denoted $z_i^b(x_{-i})$ (which depends upon x_{-i} and b), is by definition equal to

$$z_i^b(x_{-i}) := z_{-i} = b \sum_{j \neq i} x_j \tag{8}$$

Given z_{-i}, agent i now has to choose its flat contribution $x_i \in \mathbb{R}^+$, which depends upon the utility function but also upon the individual constraints. Following [Guttman, 1978], the utility of each agent i is assumed to be equal to

$$U_i(x_i, x_{-i}) = v_i(x_i + z_{-i}) - (x_i + z_{-i}) \tag{9}$$

where the function $v_i(.)$ measures the willingness-to-pay of a agent i for the public project and depends upon her own contribution but also upon the contributions of all the other agents. As in the literature on collective actions and aggregative games ([Guttman, 1978], [Cornes and Hartley, 2007] [Chen and Zeckhauser, 2018], [Cornes and Hartley, 2012], see also [Cornes, 2016] for a recent review paper), the willingness-to-pay $v_i(.)$ depends upon x_{-i} only through the sum of the contributions of the other agents, that is, z_{-i}. Given a profile of strategies $\boldsymbol{x} = (x_i, x_{-i})$, let

$$z_i = x_i + z_{-i} \tag{10}$$

be the total contribution of agent i given z_{-i} and note that the utility function of can be written as a function of the sole scalar z_i.

$$U_i(x_i, x_{-i}) = v_i(z_i) - z_i \tag{11}$$

For simplicity, we shall assume that $v_i(z_i)$ is a twice continuously differentiable increasing and concave function of z_i (with $v_i(0) = 0$) so that U_i is also a concave function[9].

[9] Concavity is actually not required. What is actually required is only the single-peakedness of U with respect to z_i, that is, the quasi-concavity of U. Such a single-peakedness assumption is fairly natural and standard (see [Guttman, 1978], see also [Greenberg and Weber, 1993]).

Regarding now the individual constraints, it can be formulated as a *budget constraint* or as a *reservation utility*.

- The budget constraint of agent i can be given as an exogenous revenue of agent i, r_i, which means that it must be the case that $z_i \leq r_i$.
- The reservation utility constraint can be given as an exogenous reservation utility of agent i, $\overline{u}_i$, which means that it must be the case that $U_i(x_i, x_{-i}) \geq \overline{u}_i$. The threshold $\overline{u}_i$ can be interpreted as a minimal quality for the public good according to agent i.

The optimization problem of a given agent i, given z_{-i}, can be formulated as a utility maximization problem subject to a budget constraint and/or subject to a reservation utility constraint. Let

- $\mathcal{B} \neq \emptyset$ be the subset of J that are subject to a budget constraint.
- $\mathcal{U} \neq \emptyset$ be the subset of J that are subject to a utility constraint.

where the $\mathcal{B}$ and $\mathcal{U}$ are assumed to form a partition of J, that is, $\mathcal{B} \cup \mathcal{U} = J$, with $\mathcal{B} \cap \mathcal{U} = \emptyset$. Agents of the group $\mathcal{U}$ may be interpreted as those agents that have other investment opportunities and are only be subject to a utility constraint. Agents of the group $\mathcal{B}$ with no other opportunities are subject to a budget constraint. Since each agent i is endowed with a utility function assumed to be a concave function of z_i, it makes thus sense to consider her *ideal contribution* z_i^* to the public project, defined independently of the participation of the other agents (i.e., for $z_{-i} = 0$). If agent i were alone, she would provide a contribution equal to z_i^*. In such a case, this also corresponds to her optimal flat contribution x_i^*.

Let $z_i^* = \arg\max_{z_i \geq 0} U_i(z_i) := v_i(z_i) - z_i$ subject to a budget utility constraint if $i \in \mathcal{B}$ and a reservation utility constraint if $i \in \mathcal{U}$. Given the assumptions on the utility function, z_i^* is unique. For the sake of interest, we assume that there exists $z_i > 0$ (with $z_i \neq z_i^*$) for which $U_i(z_i) > 0$ so that $U_i(z_i^*) \geq U_i(z_i) > 0$ for each $i \in J$.

Let $\overline{z}_i$ be the critical threshold of each agent $i \in J$ and note that agents of group $\mathcal{U}$ are characterized by two thresholds, a low one $\underline{z}_i$ and a high one $\overline{z}_i$. However, the best response of these agents only depend upon the high threshold $\overline{z}_i$.

Lemma 1 *Given $z_{-i} \geq 0$, the best response of agent of $i \in J$ is given below.*

- *If $z_{-i} \leq z_i^*$, then, $BR_i(z_{-i}) = x_i^* = z_i^* - z_{-i} > 0$*
- *If $z_{-i} \in (z_i^*, \overline{z}_i)$, then, $BR_i(z_{-i}) = x_i^* = 0$*
- *If $z_{-i} > \overline{z}_i)$ then, either $BR_i(z_{-i}) = \emptyset$ or $BR_i(z_{-i}) = 0$*

Proof. See the appendix.

The fact that $BR_i(z_{-i}) = \emptyset$ or $BR_i(z_{-i}) = 0$ depends upon the type of constraint. When $X_i(x_{-i}) = \emptyset$ for $i \in \mathcal{B}$, this simply means that this agent must pay *more* than her own revenue, which is impossible. As a result $BR_i(z_{-i}) = \emptyset$. One may interpret this as an exclusion, similar to the *violation of the no-bankruptcy condition* in aggregative games (see e.g., [Buchholz et al., 2011]). The situation is different for agents of the group $\mathcal{U}$. When $z_{-i} > \overline{z}_i$ for an agent $i \in \mathcal{U}$, her utility will be *lower* than her reservation utility so that this agent will simply reject the investment project by choosing not to contributed, i.e., $x_i = 0$.

Regarding agents of the group $\mathcal{B}$, depending upon the willingness-to-pay and the revenue, an agent i may be such that its optimal contribution z_i^* may be equal to r_i or may be lower than r_i. When $z_i^* < r_i$, the budget constraint is not binding and this occurs when z_i^* solves the unconstrained maximization of $U_i(z_i)$. When $z_i^* = r_i$, the constraint is binding. One may thus write $z_i^* = \alpha_i^* r_i$ where $\alpha_i^* \in (0, 1]$ depends upon the willingness-to-pay and the revenue of agent i. An agent $i \in \mathcal{B}$ will never provide a contribution higher than $\alpha_i^* r_i$. Consider now an agent $j \in \mathcal{U}$ with a high z_j^* and assume that when $x_i = \alpha_i^* r_i$, the best response of j, equal to $BR_j(b\alpha_i^* r_i) = z_j^* - b\alpha_i^* r_i$, is higher than r_i. In such a case, whatever the choice of agent i, agent j will always choose a contribution so high that i is left with an empty set. The following result provides a simple illustration of this when $N = 2$ but nothing is changed in the general case of an arbitrary number N of agents.

Proposition 4 *Let $b \in (0, 1]$ and $J = \{1, 2\}$ where the group $\mathcal{B} = \{1\}$ and the group $\mathcal{U} = \{2\}$. Assume that $z_1^* = r_1 < z_2^*$*

(i) A sufficient condition for the non-existence of a Nash equilibrium in individual constraints is $\frac{z_2^*}{z_1^*} > 1 + b$, that is, $z_2^* > (1 + b)r_1$.

(ii) Assume that $z_1^* \geq b\underline{z}_2$ and let $\overline{x}_1 := \frac{z_1^* - b\underline{z}_2}{(1-b^2)}$. Then, for any $\theta \in (0, 1)$, the pair of strategies $(x_1^* = \theta\overline{x}_1, x_2^* = \frac{z_1^* - \theta\overline{x}_1}{b})$ is a Nash equilibrium in shared constraint.

Proof. See the appendix.

Corollary 1 *The existence of a Nash equilibrium in endogenous shared constraint does not depend upon z_2^*. In particular, a Nash equilibrium in the game with endogenous shared constraint exists whether $z_2^* > (1 + b)r_1$ or not.*

The above corollary highlights the main difference between a Nash equilibrium in individual constraints and a Nash equilibrium in endogenous shared constraint. When agent 1 faces a budget constraint such that $z_1^* = r_1$, the existence of a Nash equilibrium in individual constraints critically depends upon the heterogeneity (or the dispersion) of the set of ideal contributions $(z_i^*)_{i \in \{1,2\}}$. If this dispersion is too high, that is, if $\frac{z_2^*}{z_1^*}$ is greater than a critical threshold, then, a Nash equilibrium in individual constraints does not exist. However, when one considers the game in shared constraint, its existence interestingly is *independent* of z_2^*. Whether z_2^* is low or very high, since agent 2 takes also into account the constraint of agent 1, its ideal contribution is irrelevant. For the Nash equilibrium in endogenous shared constraint to exist, a necessary and sufficient condition is $z_1^* \geq b\underline{z}_2$. If z_1^* is too low or if $\underline{z}_2$ is too high, then $z_1^* < b\underline{z}_2$ and there is no equilibrium in shared constraint since the underlying set K is empty. In appendix, we show that the non-emptiness of K is equivalent to the existence of a solution of the following system

$$x_1 + bx_2 \leq z_1^* \tag{12}$$

$$x_2 + bx_1 \geq \underline{z}_2 \tag{13}$$

We show in appendix that as long as $x_1 \leq \overline{x}_1$ ($\overline{x}_1$ is given as in proposition 4), the above system has a solution. For any $\theta \in (0,1)$, the pair $(x_1^* = \theta\overline{x}_1, x_2^* = \frac{z_1^* - \theta\overline{x}_1}{b})$ is a Nash equilibrium and is such that $x_1^* + bx_2^* = z_1^*$. Since for each $i =, 1, 2$, $x_i^* < z_i^*$, it thus follows that for any $\theta \in (0,1)$, the Nash equilibrium is Pareto optimal.

Corollary 2 *Each Nash equilibrium of the game with shared constraint is Pareto optimal.*

Within our particular model of collective action, when a Nash equilibrium in the game with individual constraint does not exist, there still exists a *continuum* of Nash equilibria in the game with shared constraint that are all Pareto-optimal.

Our analysis also reveals an important property of collective actions. If one only looks at the equilibrium in individual constraints when for instance $z_i^* = r_i$ for each $i \in \mathcal{B}$, the existence of such an equilibrium in individual constraints critically depends upon the dispersion of the ideal contribution $(z_i^*)_{i \in J)}$. This thus suggests that for a collective action to be possible (in individual constraints), agents of the group J must have homogenous ideal contributions, something not required for the shared constraint problem. In the general case with $N \geq 2$ agents, let

$$K = \{\boldsymbol{x} \in \mathbb{R}_+^N : x_i + b\sum_{i \neq j} x_j \leq r_i \ \forall i \in \mathcal{B} \text{ and } U_k(x_k + b\sum_{k \neq j} x_j) \geq \overline{u}_k \ \forall k \in \mathcal{U}\}$$

As in proposition 4, as long as the dispersion of $(z_i^*)_{i \in J)}$ is "too high", K will be empty and no Nash equilibrium in individual constraints exists. But in shared constraint, assuming for simplicity that $z_i^* = r_i$, the non-vacuity of K will *only* depend upon the ideal contribution of agents of $\mathcal{B}$. In case in which K is a compact and convex subset of $\mathbb{R}_+^N$, from theorem 1 ([Rosen, 1965]), a Nash equilibrium in the game with shared constraint always exist (since the utility functions are concave) while a Nash equilibrium in individual constraint might not exist.

Throughout the discussion, we made the assumption that for each $i \in \mathcal{B}$, $z_i^* = r_i$, which means that the budget constraint is always binding. It would also be possible to consider the case in which $z_i^* < r_i$. If one let $z_{i,0} := \inf_{z_i} U(z_i) = 0$, there are thus two possibilities, either $z_{i,0} < r_i$ or $z_{i,0} > r_i$. In the first case, this means that when z_{-i} is slightly higher than $z_{i,0}$, the strategy set of agent i is not empty while its utility is negative. If as in [Greenberg and Weber, 1993] however in a different framework, we allow such an agent to reject the project if its utility becomes negative, it suffices to consider the constraint c_i of agent i given as $c_i = \min\{r_i, z_{i,0}\}$ and nothing is changed.

Numerical example. Consider the following example and assume that the willingness-to-pay of a given agent i is of the form $v_i(z_i) = a_i z_i^{\frac{1}{2}} - z_i$ with $z_i \geq 0$. Without constraint, it is not difficult to show that $z_i^* = \frac{a_i^2}{4}$. To illustrate proposition 4, assume that $a_1 = 4$ and that $r_1 = 4$ so that $z_1^* = r_1 = 4$. Assume now that $b = \frac{1}{2}$ so that, from the sufficient condition in proposition 4, z_2^* should be higher than $1.5 \times 4 = 6$. Since $z_2^* = \frac{a_2^2}{4}$, this means that a_2^2 should be higher than 24. Assume that $a_2 = 6$ so that $z_2^* = 9$. Assume that the reservation utility $\overline{u}_2$ is such that $U_2(3) = \overline{u}_2$ so that $\underline{z}_2 = 3$. The shared constraint thus are

$$x_1 + \frac{1}{2}x_2 \leq 4 \tag{14}$$

$$x_2 + \frac{1}{2}x_1 \geq 3 \tag{15}$$

For $BR_2(x_1) \neq \emptyset$, it must be the case that $x_1 \leq \frac{10}{3}$. Assume that $x_1 = \frac{5}{3}$ (i.e., $\theta = \frac{1}{2}$ in proposition 4) so that $BR_2(x_1) \neq \emptyset$. Since the utility of agent 2 is increasing in x_2, agent 2 will choose x_2 such that $x_1 + \frac{1}{2}x_2 = 4$, that is $BR_2(\frac{5}{3}) = x_2^* = \frac{14}{3}$. For $x_2^* = \frac{14}{3}$, agent 1 will also chooses x_1 so that $x_1 + \frac{1}{2}x_2^* = 4$. As a result, $BR_2(\frac{14}{3}) = x_1^* = \frac{5}{3}$ and this shows that the pair $(\frac{5}{3}, \frac{14}{3})$ is a Nash equilibrium of the game with shared constraint. As long as $x_1 \leq \overline{x}_1$, equal to $\frac{10}{3}$ in this example, the game reduces to a zero-sum game since x_1 and x_2 will be chosen such that $x_1 + \frac{1}{2}x_2 = 4$

4 Conclusion

In this paper, we presented the notions of generalized games with individual constraints, generalized games with shared constraint, and generalized games with an endogenous shared constraint generated from individual constraints. We proved a simple yet interesting and rich result regarding the existence of Nash equilibria for a generalized game with an endogenous shared constraint generated from individual ones, that is the Nash equilibria of a generalized game with individual constraints are included in the set of Nash equilibria of the generalized game with endogenous shared constraint. We then studied different applications of this result, among which a public good problem and an environment control problem. A number of interesting investigations remains do be done. For instance, under which condition(s) no Nash equilibrium exist in individual constraint while it exists in endogenous shared constraint? When there is more than one equilibrium, does one of them Pareto-dominates the others?

5 Appendix

Proof of Proposition 1. The objective function of each agent is assumed to be a cost function but nothing is changed if it is instead a utility function (i.e., it sufficed to reverse the inequalities). If $\boldsymbol{x}^* = (x^{*,1}, ..., x^{*,N}) \in E$ is a Nash equilibrium for the game with individual constraints $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$, then:

- $\forall i \in J,\ x_i^* \in X_i(x_{-i}^*)$ so that $\boldsymbol{x}^* \in K$.
- $\forall i \in J,\ \forall x_i \in X_i(x_{-i}^*),\ \theta_i(x_i^*, x_{-i}^*) \leq \theta_i(x_i, x_{-i}^*)$ so $\forall i \in J,\ \forall x_i \in E_i$ such that $(x_i, x_{-i}^*) \in K,\ \theta_i(x_i^*, x_{-i}^*) \leq \theta_i(x_i, x_{-i}^*)$.

It thus follows that if the point $\boldsymbol{x}^*$ is a Nash equilibrium of the game with individual constraints $(J, E, (\theta_i)_{i \in J}, (X_i)_{i \in J})$, it is also a Nash equilibrium for the game with shared constraint $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J}))$ but the converse is however not true. If the point $\boldsymbol{x}^*$ is a Nash equilibrium of the game with shared constraint generated from the individual constraints $(J, E, (\theta_i)_{i \in J}, (K_i)_{i \in J}))$, it may be the case that the profile of strategies $\boldsymbol{x}^* \in E$ is not a Nash equilibrium of the game with individual constraint because there may exist $i \in J$ such that the best response $x_i^* := BR_i(x_{-i}^*) \in X_i(x_{-i}^*)$ is such that $\boldsymbol{x}^* \notin K$ where $\boldsymbol{x}^* = (x_i^*, x_{-i}^*)$. Such an example in which a profile of strategies which is a Nash equilibrium in shared constraint but not a Nash equilibrium in individual constraints has been given. □

Proof of proposition 2.

We shall give here a result more detailed than the one stated in the text. In what follows, we actually offer a complete characterization of the existence or non-existence of the Nash equilibrium in individual strategies. The environmental problem formulated as a game with individual constraints has a Nash equilibrium if and only if the linear system $A^T X_c = S$ (A^T denotes the transpose of A) admits at least one solution $X_c^* \in \mathbb{R}_+^N$ where $\mathrm{X}_c^* = \begin{pmatrix} c_1 x_1^* \\ c_2 x_2^* \\ \cdots \\ c_N x_N^* \end{pmatrix}$

PropositionA 2

- If the linear system $A^T X_c = S$ admits no solution in $\mathbb{R}_+^N$, then there is no Nash equilibrium.
- If the matrix A is invertible, then, there exists a unique solution X_c^* to the linear system, and if X_c^* is in $\mathbb{R}_+^N$, there exists a unique Nash equilibrium of the form:

$$\boldsymbol{x}^* = \left(x_1^* = \frac{1}{c_1} \sum_{j=1}^{N} b_{1j} S_j, ..., x_i^* = \frac{1}{c_i} \sum_{j=1}^{N} b_{ij} S_j, ..., x_N^* = \frac{1}{c_N} \sum_{j=1}^{N} b_{Nj} S_j \right)$$

  If X_c^* is not in $\mathbb{R}_+^N$, there is no Nash equilibrium.
- If the matrix A is not invertible and the linear system $A^T X_c = S$ admits at least one solution $X_c^{*,0} \in \mathbb{R}_+^N$, then the linear system admits infinitely many solutions, and there are infinitely many Nash equilibria given by

$$\begin{pmatrix} c_1 x_1^* \\ c_2 x_2^* \\ \cdots \\ c_N x_N^* \end{pmatrix} = (\mathrm{X}_c^{*,0} + Ker(A^T)) \cap \mathbb{R}_+^N$$

Proof of propositionA 2

Let's assume that this game has a Nash equilibrium $x^* = (x_1^*, ..., x_N^*)$. Such a Nash equilibrium must satisfy $\sum_j a_{ji} \times c_j \times x_j^* = S_i$ for all i. Indeed, assume that for a given i we have $\sum_j a_{ji} \times c_j \times x_j^* < S_i$, then country i could still increase its profit $f_i(x_i)$ with a x_i higher than x_i^* still satisfying the constraint, and therefore x^* would not be a Nash equilibrium. Therefore a Nash equilibrium $x^* = (x_1^*, ..., x_N^*)$ for this problem with individual constraints must satisfy: $A^T X_c^* = S$ with $X_c^* \in \mathbb{R}_+^N$, with

$$\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1N} \\ a_{21} & a_{22} & ... & a_{2N} \\ ... & ... & ... & ... \\ a_{N1} & a_{N2} & ... & a_{NN} \end{pmatrix}, \mathrm{X}_c^* = \begin{pmatrix} c_1 x_1^* \\ c_2 x_2^* \\ ... \\ c_N x_N^* \end{pmatrix}, S = \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix}$$

And this has a solution if and only if the linear system $A^T X_c = S$ has a solution $X_c^* \in \mathbb{R}_+^N$.

- If A is invertible, we have an explicit formula for the Nash equilibrium candidate vector since:

$$\begin{pmatrix} c_1 x_1^* \\ c_2 x_2^* \\ ... \\ c_N x_N^* \end{pmatrix} = (\mathrm{A}^T)^{-1} \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix}$$

If we denote $B = (A^T)^{-1}$, we have:

$$\begin{pmatrix} c_1 x_1^* \\ c_2 x_2^* \\ ... \\ c_N x_N^* \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & ... & b_{1N} \\ b_{21} & b_{22} & ... & b_{2N} \\ ... & ... & ... & ... \\ b_{N1} & b_{N2} & ... & b_{NN} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ ... \\ S_N \end{pmatrix}$$

And for all i $x_i^* = \frac{1}{c_i} \sum_{j=1}^{N} b_{ji} S_j$, and we have a unique Nash equlibrium candidate $x^* = (x_1^* = \frac{1}{c_1} \sum_{j=1}^{N} b_{1j} S_j, ..., x_i^* = \frac{1}{c_i} \sum_{j=1}^{N} b_{ij} S_j, ..., x_N^* = \frac{1}{c_N} \sum_{j=1}^{N} b_{Nj} S_j)$.

- If A is not invertible and the linear system $A^T X_c = S$ admits one solution $X_c^{*,0} \in \mathbb{R}_+^N$, then $A^T X_c = A^T X_c^{*,0}$, which is equivalent to $A^T(X_c - X_c^{*,0}) = 0$, which is equivalent to $X_c = (X_c^{*,0} + Ker(A^T)) \cap \mathbb{R}_+^N$, and there are infinitely many Nash equilibria.

- If the linear system $A^T X_c = S$ admits no solution in $\mathbb{R}_+^N$, then there is no Nash equilibrium.

□

Proof of proposition 3

It is easy to see that thea game always satisfies the assumptions of the existence result of Rosen for n-person concave games (see theorem 1 in the text). Indeed, the payoff functions $f_i(x_i)$ are concave, and the shared constraint space K is clearly a convex compact space:

$$K = \{\boldsymbol{x} \in \mathbb{R}^N_+ : g_i(\boldsymbol{x}) \leq S_i, i = 1, 2, ..., N\} \tag{16}$$

$$K = \{\boldsymbol{x} \in \mathbb{R}^N_+ : a_{ii} \times c_i \times x_i + \sum_{j \neq i} a_{ji} \times c_j \times x_j \leq S_i, i = 1, 2, ..., N\} \tag{17}$$

Therefore, there always exist a Nash equilibrium for the game with shared constraint. □

Proof of lemma 1 To avoid confusion between agents of $\mathcal{B}$ and agents of $\mathcal{U}$, the first ones will be indexed by j while the other ones will be indexed by k.

Consider first agents indexed by j that belong to $\mathcal{B}$. Let $z_{-j} = b \times (\sum_{i \neq j} x_i)$ be the sum of contributions of the agents in $\mathcal{B}$ except agent j and let $\overline{z}_j := r_j > 0$ be a critical threshold. It is assumed that $U_j(r_j) > 0$.

If $z_{-j} > \overline{z}_j$, then, $X_j(z_{-j}) = \emptyset$. Otherwise, that is, if $z_{-j} \leq \overline{z}_i$, $X_i(z_{-j}) \neq \emptyset$. For each agent $j \in \mathcal{B}$, her ideal contribution is equal to $z^*_j \in (0, \overline{z}_j)$ so that the best response of agent $j \in \mathcal{B}$ given z_{-j} is as follows.

- If $z_{-j} \leq z^*_j$, then, $BR_j(z_{-j}) = x^*_j = z^*_j - z_{-j} > 0$
- If $z_{-j} \in (z^*_j, \overline{z}_j)$, then, $BR_j(z_{-j}) = x^*_j = 0$
- If $z_{-j} > \overline{z}_j$ then, $BR_j(z_{-j}) = \emptyset$

Consider now agents $k \in \mathcal{U}$. As before, let $z_{-k} := b \times \sum_{k \neq i} x_i$. For agents $k \in \mathcal{U}$, given the reservation utility $\overline{u}_k$, since U_k is concave and since by assumption $z^*_k > 0$ is such that $U_k(z^*_k) > 0$, there thus exists two critical thresholds, a low one defined as $\underline{z}_k := \inf_{z_k} U_k(z_k) = \underline{u}_k$ and a high one defined as $\overline{z}_k := \underline{z}_k := \sup_{z_k} U_k(z_k) = \underline{u}_k$. It thus follows that if $z_{-k} > \overline{z}_k$, agent k rejects the project so that $X_k(z_{-k}) = \emptyset$. If $z_{-k} < \underline{z}_k$, agent will choose her flat contribution x_k so that $z_k = z^*_k$. Noting that for each agent $k \in \mathcal{U}$, $z^*_k \in (\underline{z}_k, \overline{z}_k)$, the best response of agent of $k \in \mathcal{U}$ is given below.

- If $z_{-k} \leq z^*_k$, then, $BR_j(z_{-k}) = x^*_k = z^*_k - z_{-k} > 0$
- If $z_{-k} \in (z^*_k, \overline{z}_k)$, then, $BR_j(z_{-k}) = x^*_k = 0$
- If $z_{-k} > \overline{z}_k)$ then, $BR_k(z_{-k}) = 0$

and this concludes the proof □

Proof of proposition 4

Part i). It is assumed that $\alpha^*_1 = 1$ so that $z^*_1 = \alpha^*_1 r_1 = r_1$. When $x_1 = z^*_1$, if $x^*_2 := BR_2(b \times z^*_1) > z^*_1$, then, $X_1(x^*_2) = \emptyset$ so that No Nash equilibrium in individual constraints can exist. From lemma 1, $BR_2(b \times z^*_1) = z^*_2 - b \times z^*_1$ and $z^*_2 - b \times z^*_1 > z^*_1$ is equivalent to $z^*_2 > z^*_1(1 + b)$. When this is the case, a Nash equilibrium in individual constraints does not exist □ Part ii). Let $K = \{(x_1, x_2) \in \mathbb{R}^+_2 : x_1 + bx_2 \leq z^*_1$ and $U_2(x_1, x_2) \geq \overline{u}_2\}$. For a Nash equilibrium with shared constraint to exist, we must first prove the non-vacuity of K, which in turn is equivalent to the existence of a solution that solves the following system.

$$x_1 + bx_2 \leq z_1^* \tag{18}$$
$$U_2(x_2 + bx_1) \geq \overline{u}_2 \tag{19}$$

We know from lemma 1 that there exists two critical thresholds $\underline{z}_2$ and $\overline{z}_2$ defined as $\underline{z}_2 = \inf_{z_2} U_2(z_2) = \overline{u}_2$ and $\underline{z}_2 = \sup_{z_2} U_2(z_2) = \overline{u}_2$. It thus follows that equation (19) is equivalent to $z_2 \in [\underline{z}_2, \overline{z}_2]$, where $z_2 = x_2 + bx_1$. Since we already know that for z_2^*, an equilibrium does not exist, one must search for a solution $z_2 \in [\underline{z}_2, z_2^*)$. For K to be non-empty, it thus suffices to find a solution to the following system

$$x_1 + bx_2 \leq z_1^* \tag{20}$$
$$x_2 + bx_1 \geq \underline{z}_2 \tag{21}$$

Solving equation (20) in x_2 yields $x_2 \leq \frac{z_1^* - x_1}{b}$ while solving equation (21) yields $x_2 \geq \underline{z}_2 - bx_1$. For a solution in x_2 to exist, $\frac{z_1^* - x_1}{b}$ must be higher than $\underline{z}_2 - bx_1$. Solving $\frac{z_1^* - x_1}{b} \geq \underline{z}_2 - bx_1$ yields $z_1^* - b\underline{z}_2 \geq x_1(1 - b^2)$. There thus exists a solution in x_2 iff $z_1^* - b\underline{z}_2 \geq 0$. Assuming that $z_1^* - b\underline{z}_2 \geq 0$, x_1 must be such that $x_1 \in \left[0, \frac{z_1^* - b\underline{z}_2}{(1-b^2)}\right]$. Let $\overline{x}_1 := \frac{z_1^* - b\underline{z}_2}{(1-b^2)}$ be the maximal value of x_1 and assume that $x_1 = \theta\overline{x}_1$ for some $\theta \in (0, 1)$. By construction, $BR_2(\theta\overline{x}_1) \neq \emptyset$. Consider now the best response of agent 2, $BR_2(\theta\overline{x}_1) = x_2^*$. Agent 2 will choose the highest contribution subject to the constraint given by equations (20) and (21). Agent 2 will thus choose x_2^* so that $\theta\overline{x}_1 + bx_2^* = z_1^*$, which means that $x_2^* = \frac{z_1^* - \theta\overline{x}_1}{b}$. Given $x_2^* = \frac{z_1^* - \theta\overline{x}_1}{b}$, agent 1 will choose x_1 such that $x_1 + bx_2^* = z_1^*$ so that $BR_1(\frac{z_1^* - \theta\overline{x}_1}{b}) = x_1^* = \theta\overline{x}_1$. For any $\theta \in (0, 1)$, the pair $(\theta\overline{x}_1, \frac{z_1^* - \theta\overline{x}_1}{b})$ is a Nash equilibrium in shared constraint $\square$

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