Valuing an investment project using no-arbitrage and the alpha-maxmin criteria: From Knightian uncertainty to risk[*]

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Abstract

We consider a two-period irreversible investment decision problem in which the firm can either invest in period 0 or in period 1. The firm is assumed to be able to specify a set of three scenarios or more but not a probability measure. Assuming the option to wait is valued with the no-arbitrage principle, when the firm makes use of the criteria α-maxmin, we show the firm ends up with a known probability measure that assigns a positive probability to four (or three) scenarios only.

Keywords: Knightian uncertainty, investment decision, option to wait, no-arbitrage, α-maxmin
JEL Codes. D81, G11.

1 Introduction

Consider an irreversible investment decision problem in which the firm can either invest in period 0 or delay the decision in period 1. In a situation of risk, when the relevant probability measure is perfectly known, the firm should invest in period 0 only if the net present value is higher than the value of the option to wait. In [Dixit et al., 1994] chapter 2, they consider such a two period case in which the project's value can increase with a known probability p or decrease with the complementary probability. In [Nishimura and Ozaki, 2007], they consider a similar example but in which the firm is uncertain about her estimation of p, the probability of the good scenario (called boom) in period 1. However, the firm is still assumed to be able to specify a set of plausible values of p.

The aim of this paper is to reconsider this two-period irreversible investment decision problem. We assume that the firm is able to specify a (finite) set of scenarios but is unable to specify a probability measure. Such a situation is usually called Knightian uncertainty (see [Etner et al., 2012] for a recent review on the subject). Assuming the option to wait is valued using the no-arbitrage principle when the firm uses, as in [Schröder, 2011] and [Gao and Driouchi, 2013], the α-maxmin criteria, we show that the firm ends up with a known probability measure that assigns a positive probability to only four (possibly three) scenarios only, all the others being irrelevant. Everything is thus as if the decision problem under uncertainty were reduced to decision problem under risk.

The paper is organized as follows. In section 2, we present the investment decision problem and in section 3, we present the analysis of the problem under Knightian uncertainty.

2   The investment decision problem

We consider a two-period model similar to [Dixit et al., 1994] in which the firm has the possibility to invest at period 0 or at period 1 when the uncertainty is resolved. Let I be the cost the project. When the firm invests, in period 0 or 1, it can produce forever (at zero marginal cost) one unit of a good sold at the market price. The current price of the good P_0 is known but the price in period 1 is unknown. After period 1, the price is assumed to remain constant, i.e., $P_t = P_1$ for $t = 1, 2, ...\infty$. Let

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\} \quad n \geq 3 \tag{1}$$

be the set of scenarios (or state of the world) regarding the price in period 1 defined as

$$P_1(\omega) = \omega P_0 \quad \omega \in \Omega \tag{2}$$

where for simplicity we assume that $\omega_1 < \omega_2 < ... < \omega_n$. The actual state is observed in period 1 *before* the firm makes her decision. Let $r>0$ be the risk-free rate. Since there is no longer uncertainty, all the cash-flows are discounted at the risk-free rate r. The value of the project seen from period 1 in scenario $\omega \in \Omega$ is equal to

$$V_1(\omega) = \sum_{t=1}^{\infty} \frac{P_1(\omega)}{(1+r)^{t-1}} = \omega \underbrace{\left(\frac{P_0}{r}\right)}_{V_0}(1+r) = \omega \, V_0(1+r) \tag{3}$$

The firm will invest in the project if $V_1(\omega) \geq I$ and reject it if $V_1(\omega) < I$. The value of the project thus is equal to

$$\Pi_1(\omega) := \max\{V_1(\omega) - I; 0\} \tag{4}$$

and is identical to the payoff of a call option with maturity $T = 1$, strike price I and underlying asset V_0. For the sake of interest, we assume that $\omega_1 V_0(1 + r) < I$ and that $\omega_n V_0(1 + r) > I$.

When the firm invests at time $t = 0$, the cash-flow in period 0 is equal to $P_0 - I$ (which is typically negative) and is equal to the (positive) random variable $V_1(\omega)$ in period 1.

Let $\mathbb{P}$ be the objective (or historical) probability measure such that

$$\mathbb{P}(\{\omega_i\}) > 0 \quad i = 1, 2, ..., n. \tag{5}$$

Risk is the situation in which the firm explicitly knows $\mathbb{P}$ while uncertainty is the situation in which $\mathbb{P}$ is not perfectly known.

3 Investment decision problem under risk and uncertainty

Consider first the case of risk and assume that the firm is risk-neutral. Since the underlying probability measure $\mathbb{P}$ is known, the value of investing at time $t = 0$ is equal to

$$P_0 + \frac{\mathbb{E}^{\mathbb{P}}(V_1(\omega))}{1 + r} - I \tag{6}$$

while the present value of investing at time $t = 1$ is equal to

$$\frac{\mathbb{E}^{\mathbb{P}}\Pi_1(\omega)}{1 + r} \tag{7}$$

It is optimal to invest in period 0 (respectively period 1) if equation (6) is strictly higher (respectively strictly lower) than equation (7).

Consider now the case of Knightian uncertainty in which the probability measure is unknown so that no expectation can be computed. When the objective probability measure $\mathbb{P}$ (assuming it exists) is unknown, a decision-maker (such as a Bayesian statistician) might be able to form an unique (subjective) probability measure $\mathbb{P}_{Sub}$ over the state of the world Ω so that the problem reduces to risk. However, specifying such an unique probability measure may be too demanding in general. A given decision-maker may only be able to specify a set $\mathcal{P}$ of probability measures over Ω, and this approach has been called multiple priors in the economics literature, see e.g., [Gilboa et al., 2008] for a very readable discussion and references therein. Interestingly, in the theory of arbitrage-free securities markets, developed among others by [Harrison and Kreps, 1979], [Taqqu and Willinger, 1987] for the case of a finite Ω, the knowledge of the objective probability measure is irrelevant. As clearly stressed by [Taqqu and Willinger, 1987], "investors may disagree on their choice of $\mathbb{P}$ but they all agree on what states of nature are possible", which means that $\mathcal{P}$ represents the set of *all* the probability measures $\mathbb{P}$ over Ω such that $\mathbb{P}(\{\omega_i\}) > 0$ for $i = 1, 2, ..., n$.

Assuming $K \geq 1$ risky securities and one default risk-free asset, [Taqqu and Willinger, 1987]

show (see theorem 3.1) that no arbitrage[1] is equivalent to the existence (but not necessarily unique-ness) of a probability measure $\mathbb{Q} := (\mathbb{Q}(\{\omega_i\})_{i=1}^n$ called an *equivalent martingale measure*. This kind of result is known as the fundamental theorem of asset pricing (see also [Cerreia-Vioglio et al., 2015] section 2.2 for a presentation). The measure $\mathbb{Q}$ is said to be equivalent to $\mathbb{P}$ since $\mathbb{Q}(\{\omega_i\}) > 0$ is equivalent to $\mathbb{P}(\{\omega_i\}) > 0$ for each $i = 1, ..., n$ and it is said to be a martingale measure (within our framework) since $\frac{\mathbb{E}^{\mathbb{Q}}(V_1(\omega)|V_0)}{1+r} = V_0$. To find a martingale measure, [Taqqu and Willinger, 1987] rely on duality theory for linear programming. Once the set of martingale measures $\mathbb{Q}$ is known (see [Pliska, 1997] p. 12 example 1.2 for a numerical application), the value of an option seen from $t = 0$ is computed as a discounted expectation of the payoff under a (given) measure $\mathbb{Q}$, that is,

$$\frac{\mathbb{E}^{\mathbb{Q}}(\Pi_1(\omega))}{1+r} \tag{8}$$

The analysis of no arbitrage is difficult even when Ω is finite since it requires to use a version of the Hahn-Banach theorem, i.e., the separating hyperplane theorem (see [Elliott and Kopp, 2004] chapter 3 or [Pliska, 1997] p. 14 for a discussion). In this paper, we follow the straightforward approach introduced in [Braouezec and Grunspan, 2016], which only requires to locate two points on a quadrilateral (or a triangle) to find the set of arbitrage-free prices of a *given* option. This approach is very simple but is less general than the classical one (e.g., [Taqqu and Willinger, 1987]) since it considers a given option and not all the possible contingent claims one can think of.

3.1 No-arbitrage and the α-maxmin criteria

Let $\mathbf{P}_i := (V_1(\omega_i), \Pi_1(\omega_i)) \in \mathbb{R}_+^2$ for $i = 1, 2, ..., n$ be a set of planar points and let us call Γ be the convex hull[2] spanned by the set of n points $\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n$, that is

$$\Gamma := \text{Conv}\{\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n\} \tag{9}$$

Since the n points form a set of planar points, Γ is just a *convex polygon*. In the generic case, there are some states of the world $\omega_i \in \Omega$ such that $V_1(\omega_i) > I$ and some others for which $V_1(\omega_i) < I$. By definition, a point $\mathbf{P}_i$ which is not a vertex of Γ can be expressed as a convex combination of the vertices. As a result, this point $\mathbf{P}_i$ is irrelevant. In figure 1, the convex hull is a *quadrilateral*[3] so that it has four vertices, namely $\mathbf{P}_1$, $\mathbf{P}_k$, $\mathbf{P}_{k+1}$ and $\mathbf{P}_n$. Let F_0 be the point defined as follows

$$F_0 := ((1+r)V_0; (1+r)\Pi_0) \tag{10}$$

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[1]See also [Varian, 1987] for a simple presentation of the formalization of the no-arbitrage condition. [Varian, 1987] uses the notion of a vector of state prices rather than the notion of a martingale measure but these two concepts are identical up to the discount factor, i.e., $\frac{1}{1+r}$.

[2]The convex hull is the smallest convex set that contains the n points.

[3]If there exists $\omega_i \in \Omega$ such that $\omega_i V_0 = I$, then Γ reduces to a triangle.

Figure 1: The convex hull Γ is a quadrilateral

where Π_0 is the value of the option at time $t = 0$. Recall that an arbitrage opportunity is a situation in which it is possible to design a costless portfolio in period 0 such that its value in period 1 is strictly positive in some state(s) of the world and zero otherwise. Let $\text{Int}(\Gamma)$ be the interior of the convex polygon Γ and $\partial\Gamma$ be its boundary. The following result is proved in [Braouezec and Grunspan, 2016].

Proposition 1 *The value of the call option Π_0 is arbitrage-free if and only if $F_0 \in \text{Int}(\Gamma)$.*

The idea behind this result is simple. If $F_0 \notin \text{Int}(\Gamma)$, then it becomes possible to separate by a line v_1 whose equation is $\Upsilon_1(\omega_i) = aV_1(\omega_i) + b$ the point F_0 from the interior of Γ. If the option is overvalued, i.e., $\Pi_0 > (1 + r)\overline{\Pi}_0$, this means that $\Upsilon_1(\omega) > \Pi_1(\omega)$ for each $\omega \in \Omega$ and this yields an elementary arbitrage opportunity. Let $\overline{\Pi}_0$ and $\underline{\Pi}_0$ be two critical option values such that the points $\overline{F}_0 = ((1 + r)V_0; (1 + r)\overline{\Pi}_0))$ and $\underline{F}_0 = ((1 + r)V_0; (1 + r)\underline{\Pi}_0))$ lie on the boundary of Γ, see Fig. 1. The methodology also works in a multiperiod framework but is more challenging since the problem is then equivalent to determine a sequence of convex hulls (and not only one). In [Braouezec and Grunspan, 2016], they explicitly solve the valuation problem of an American option in a two period trinomial model.

From the definition of $\overline{\Pi}_0$ and $\underline{\Pi}_0$, as long as $\Pi_0 \in (\underline{\Pi}_0, \overline{\Pi}_0)$, it is arbitrage-free. Following

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[Gao and Driouchi, 2013] or [Schröder, 2011], the firm is assumed to use the α-maxmin criteria defined as

$$\alpha \overline{\Pi}_0 + (1 - \alpha) \underline{\Pi}_0 \tag{11}$$

which leads to an arbitrage-free option value since $\alpha \in (0, 1)$. It is usual to interpret α as a measure of the degree of optimism of the firm (see e.g., [Etner et al., 2012]).

Proposition 2 *Assume that $\omega_k < \frac{I}{V_0(1+r)} < 1$ and $\omega_{k+1} > 1$. For a given $\alpha \in (0, 1)$, the probability measure $\mathbb{Q}^\alpha := (\mathbb{Q}^\alpha(\{\omega_i\}))_{i=1}^n$ defined as*

$$\mathbb{Q}^\alpha(\{\omega_1\}) = \alpha \left(\frac{\omega_n - 1}{\omega_n - \omega_1} \right) > 0 \qquad \mathbb{Q}^\alpha(\{\omega_k\}) = (1 - \alpha) \left(\frac{\omega_{k+1} - 1}{\omega_{k+1} - \omega_k} \right) > 0$$

$$\mathbb{Q}^\alpha(\{\omega_{k+1}\}) = (1 - \alpha) \left(\frac{1 - \omega_k}{\omega_{k+1} - \omega_k} \right) > 0 \quad \mathbb{Q}^\alpha(\{\omega_n\}) = \alpha \left(\frac{1 - \omega_1}{\omega_n - \omega_1} \right) > 0$$

$$\mathbb{Q}^\alpha(\{\omega_i\}) = 0 \qquad \text{for } \omega_i \notin \{\omega_1, \omega_k, \omega_{k+1}, \omega_n\}$$

is such that

$$\frac{\mathbb{E}^{\mathbb{Q}^\alpha} \Pi_1(\omega)}{1 + r} = \alpha \overline{\Pi}_0 + (1 - \alpha) \underline{\Pi}_0 \tag{12}$$

Proof. See the appendix.

To understand the intuition behind this proposition, recall that $\underline{\Pi}_0$ is the lower bound of the option value. In appendix, we show that this lower bound of the option value, $\underline{\Pi}_0$, can be written as

$$\underline{\Pi}_0 = \frac{\mathbb{E}^{\underline{\mathbb{Q}}} \Pi_1(\omega)}{1 + r} \tag{13}$$

which can be interpreted as the expected discounted value of the option payoff under a probability measure $\underline{\mathbb{Q}}$ that assigns a positive weight to the states ω_k and ω_{k+1} only. Everything is as if $\underline{\Pi}_0$ were computed using a binomial model with the two states of the world ω_k and ω_{k+1}. In the same vein, the upper bound of the option value, $\overline{\Pi}_0$, can be written as

$$\overline{\Pi}_0 = \frac{\mathbb{E}^{\overline{\mathbb{Q}}} \Pi_1(\omega)}{1 + r} \tag{14}$$

which can once again be interpreted as the expected discounted value of the option payoff under a probability measure $\overline{\mathbb{Q}}$ that assign positive weight to the states ω_1 and ω_n only.

Consider now the probability measure $\mathbb{Q}^\alpha$ defined as a convex combination of the two probability measures $\overline{\mathbb{Q}}$ and $\underline{\mathbb{Q}}$ (written as column vectors), that is, $\mathbb{Q}^\alpha = \alpha \overline{\mathbb{Q}} + (1 - \alpha) \underline{\mathbb{Q}}$ and note that $\mathbb{Q}^\alpha$ assigns a positive weight to the states of the world $\omega_1, \omega_k, \omega_{k+1}$ and ω_n only. By making use of the linearity property of the expectation operator, it is not difficult to show that equation (12) holds.

This result shows interestingly that while the firm is completely unaware of the true probability measure $\mathbb{P}$, by using the no-arbitrage principle and the α-maxmin criteria, she ends up with an explicit set of weights $\mathbb{Q}^{\alpha}$ that can be interpreted as a probability measure (which is a martingale measure).

Corollary 1 *Everything is as if the firm were in a situation of risk equipped with the probability measure $\mathbb{Q}^{\alpha}$ that assigns a positive weight to the four states of the world $\omega_1, \omega_k, \omega_{k+1}$ and ω_n only. As a result, the probability measures $\mathbb{Q}^{\alpha}$ and $\mathbb{P}$ are not equivalent, i.e., for some $i \in \{1, 2, ..., n\}$, $\mathbb{Q}^{\alpha}(\{\omega_i\}) = 0$ while $\mathbb{P}(\{\omega_i\}) > 0$.*

The fact that the probability measures $\mathbb{Q}^{\alpha}$ and $\mathbb{P}$ are not equivalent reflects the fact that we only value a *given* option with the no-arbitrage principle and not the consequence of no-arbitrage in general. Under the (additive) probability measure $\mathbb{Q}^{\alpha}$, we obtain a partition of Ω in two subsets, $\mathcal{N}$ and $\mathcal{E}^*$, that are respectively, to use the terminology introduced in [Chateauneuf et al., 2007], the set of states of the world that cannot occur (the $\mathbb{Q}^{\alpha}$-null events) and the set states of the world that can occur with a positive probability strictly lower than one. We are now in a position to explicitly derive the optimal investment decision rule using the probability measure $\mathbb{Q}^{\alpha}$.

Proposition 3 *Whether the firm is risk-averse or not, the optimal investment decision rule is as follows. Invest in period 0 if $V_0 - I > \frac{\mathbb{E}^{\mathbb{Q}^{\alpha}} \Pi_1(\omega)}{1+r} - P_0$ and postpone if the inequality is reversed.*

Proof. See the appendix

The optimal investment decision rule does not depend upon the characteristics of the decision maker such as risk-aversion because the two critical option prices $\underline{\Pi}_0$ and $\overline{\Pi}_0$ are determined using no-arbitrage.

3.2   No-arbitrage and the α-maxmin expected utility criteria

Since the probability measure $\mathbb{P}$ is unknown, no expected utility can be computed under $\mathbb{P}$. However, we have seen that the application of the no-arbitrage principle to value the option to wait led us to two pricing measures $\underline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}$ under which the critical price $\underline{\Pi}_0$ and $\overline{\Pi}_0$ can be thought of as an expected discounted value of the payoff (equations (13) and (14)). Let $\delta < 1$ be the subjective discount factor used to compute the expected utility of the form $\mathbb{E}[\delta U(.)]$ and assume indeed that $\delta = \frac{1}{1+r}$. In line with [Ghirardato et al., 2004], the α-maxmin expected utility criteria can be written as

$$\alpha \frac{\mathbb{E}^{\overline{\mathbb{Q}}} U(\Pi_1(\omega))}{1+r} + (1-\alpha) \frac{\mathbb{E}^{\underline{\mathbb{Q}}} U(\Pi_1(\omega))}{1+r} \tag{15}$$

where the utility function $U(x)$ is assumed to be an increasing and continuous function of x. Equation (15) is closely related to equation (12) of proposition 19 of [Ghirardato et al., 2004] but they are not, strictly speaking, identical since $\underline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}$ do *not* minimize and maximize the expected utility $\mathbb{E}^{\overline{\mathbb{Q}}}U(.)$ and $\mathbb{E}^{\underline{\mathbb{Q}}}U(.)$ respectively over the set of relevant probability measures. Equation (15) takes as given the two measures $\underline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}$. It is not difficult to show that equation (15) is equal to $\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}U(\Pi_1(\omega))}{1+r}$, so that

$$\mathbb{E}^{\mathbb{Q}^{\alpha}}U(\Pi_1(\omega)) = \alpha\mathbb{E}^{\overline{\mathbb{Q}}}U(\Pi_1(\omega)) + (1-\alpha)\mathbb{E}^{\underline{\mathbb{Q}}}U(\Pi_1(\omega)) \tag{16}$$

Assume now that the option price Π_0 is equal to the certainty equivalent[4] Π_0^C defined as

$$\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}U(\Pi_1(\omega))}{1+r} = U(\Pi_0^C) \iff \Pi_0^C = U^{-1}\left(\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}U(\Pi_1(\omega))}{1+r}\right) \tag{17}$$

When $U(x)$ is linear, Π_0^C reduces to the right-hand side of equation (12), that is, to the α-maxmin. However, when $U(x)$ is a concave function of x, the option price will be *lower* than the α-maxmin. In [Musiela and Zariphopoulou, 2004], they note that "no linear pricing mechanism can be compatible with the concept of utility based valuation". While this is not incorrect, a version of the expected utility criteria can still be used to choose an option price Π_0 as long as $\Pi_0 \in (\underline{\Pi}_0, \overline{\Pi}_0)$, i.e., it is arbitrage-free. This means that if $\Pi_0^C < \underline{\Pi}_0$, the chosen price *can not* be equal to the certainty equivalent. This situation in which $\Pi_0^C < \underline{\Pi}_0$ occurs for instance when the decision-maker is risk-averse (U concave) and pessimistic ($\alpha = 0$) but can also occur when the decision-maker is very risk-averse even when $\alpha > 0$. Since the option price Π_0 should be arbitrage-free, it must defined as follows

$$\Pi_0 = \max\{\underline{\Pi}_0; \Pi_0^C\} \tag{18}$$

Strictly speaking, since $\underline{\Pi}_0$ is not an arbitrage-free option price, in case $\Pi_0^C < \underline{\Pi}_0$, the option price Π_0 should be equal to $\underline{\Pi}_0 + \epsilon$ for any arbitrarily small $\epsilon > 0$.

3.3 Numerical example

Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ where $\omega_1 = 0.8$, $\omega_2 = 0.85$, $\omega_3 = 1.05$, $\omega_4 = 1.1$ be the four relevant states of the world. Let $P_0 = 1$, $r = 2\%$ so that $V_0 = \frac{1}{0.02} = 50$. Assume that $I = 45$ so that $V_0 - I = 5$. Let $\alpha = \frac{1}{2}$ and $\mathbb{Q}^{\frac{1}{2}}(\{\omega_i\}) = q_i$. Using proposition 2, $q_1 = 0.166$, $q_2 = 0.125$, $q_3 = 0.375$ and $q_4 = 0.333$ and note that $\Pi_1(\omega_1) = \Pi_1(\omega_2) = 0$ and that $\Pi_1(\omega_3) = 8.55$ and $\Pi_1(\omega_4) = 11.1$. It is easy to show that $\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}\Pi_1(\omega)}{1+r} = 6.76$. Note that $\underline{\Pi}_0 = 6.286$ and that $\overline{\Pi}_0 = 7.254$ so that $(0.5 \times 6.286) + (0.5 \times 7.254)$ also yields 6.76. Since $V_0 - I = 5$ and $\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}\Pi_1(\omega)}{1+r} - P_0 = 5.76$. From proposition 3, it is optimal to wait.

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[4]See [Staum, 2007] for a review of existing option pricing methods based on expected utility.

Assume now that $U(x) = \sqrt{x} = x^{0.5}$ for $x \geq 0$. In this example, it is not difficult to show that $\mathbb{E}^{\mathbb{Q}^{\alpha}} U(\Pi_1(\omega)) = q_3 U(\Pi_1(\omega_3)) + q_4 U(\Pi_1(\omega_4))$ which is equal to $\frac{0.375\sqrt{8.55} + 0.333\sqrt{11.1}}{1.02} = 2.16$ so that $\Pi_0^C = 2.206^2 = 4.67$. Since the certainty equivalent is lower than 6.286, the chosen price Π_0 must be equal to $\underline{\Pi}_0 = 6.286$ plus $\epsilon > 0$, say $\Pi_0 = 6.29$. Consider now the utility function defined as $U(x) = x^{0.9}$ for $x \geq 0$, i.e., the decision-maker is less risk-averse. The (discounted) expected utility is equal to 5.35 so that the certainty equivalent is now equal to 6.13. Since it is higher than 6.286, the option price Π_0 can now be equal to the certainty equivalent.

4 Appendix: proofs

Notation. If $\boldsymbol{x} = (x_1; x_2)$ is a row vector, its transpose denoted $\boldsymbol{x}'$ is a column vector, that is $\boldsymbol{x}' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Proof of proposition 2. It is assumed that $\omega_{k+1} > 1$ and $\omega_k < \frac{I}{V_0(1+r)}$. The point defined as $\underline{\mathbf{F}}_0 = ((1+r)V_0; (1+r)\underline{\Pi}_0)$ lies on the boundary of Γ and is located on the segment formed by $\mathbf{P}_k = (\omega_k(1+r)V_0; \Pi_1(\omega_k))$ and $\mathbf{P}_{k+1} = (\omega_{k+1}(1+r)V_0; \Pi_1(\omega_{k+1})$, see Fig 1. Since $\underline{\mathbf{F}}_0'$ lies on the segment formed by $\mathbf{P}_k'$ and $\mathbf{P}_{k+1}'$, there exists a weight $\underline{q} \in (0,1)$ such that $\underline{\mathbf{F}}_0' = \underline{q}\mathbf{P}_{k+1}' + (1-\underline{q})\mathbf{P}_k'$. This leads to the following linear system of two equations with two unknowns

$$(1+r)V_0 = \underline{q}\omega_{k+1}(1+r)V_0 + (1-\underline{q})\omega_k(1+r)V_0 \tag{19}$$

$$(1+r)\underline{\Pi}_0 = \underline{q}\Pi_1(\omega_{k+1}) + (1-\underline{q})\Pi_1(\omega_k) \tag{20}$$

which leads to

$$\underline{q} = \frac{1-\omega_k}{\omega_{k+1} - \omega_k} \quad \text{and} \quad 1 - \underline{q} = \frac{\omega_{k+1} - 1}{\omega_{k+1} - \omega_k} \tag{21}$$

Equation (20) can be written as $\underline{\Pi}_0 = \frac{\underline{q}\Pi_1(\omega_{k+1}) + (1-\underline{q})\Pi_1(\omega_k)}{1+r}$ and can also be interpreted as the expected discounted value of the payoff under a probability measure $\underline{\mathbb{Q}} = (0, ..., 0, 1 - \underline{q}, \underline{q}, 0, ..., 0)$ that only assign positive weight to the states ω_k and ω_{k+1}, that is,

$$\underline{\Pi}_0 = \frac{\mathbb{E}^{\underline{\mathbb{Q}}} \Pi_1(\omega)}{1+r} \tag{22}$$

Since $\Pi_1(\omega_k) = 0$ and $\Pi_1(\omega_{k+1}) = \omega_{k+1}(1+r)V_0 - I$, it follows that

$$\underline{\Pi}_0 = \frac{\mathbb{E}^{\underline{\mathbb{Q}}} \Pi_1(\omega)}{1+r} = \frac{\underline{q}(\omega_{k+1}(1+r)V_0 - I)}{1+r} \tag{23}$$

A similar analysis can be done for the points $\mathbf{P}_1'$ and $\mathbf{P}_n'$ and the point $\overline{\mathbf{F}}_0'$, where $\overline{\mathbf{F}}_0 = ((1+r)V_0; (1+r)\overline{\Pi}_0)$ since the point $\overline{\mathbf{F}}_0'$ is located on the segment formed by the points $\mathbf{P}_1'$ and $\mathbf{P}_n'$. As before, there exists $\overline{q} \in (0,1)$ such that $\overline{\mathbf{F}}_0' = \overline{q}\mathbf{P}_n' + (1-\overline{q}_n)\mathbf{P}_1'$ which leads to

$$\overline{q} = \frac{1-\omega_1}{\omega_n - \omega_1} \quad \text{and} \quad (1-\overline{q}) = \frac{\omega_n - 1}{\omega_n - \omega_1} \tag{24}$$

As before, $\overline{\Pi}_0 = \frac{\overline{q}\Pi_1(\omega_n)+(1-\overline{q})\Pi_1(\omega_1)}{1+r}$. Since $\Pi_1(\omega_1) = 0$ and $\Pi_1(\omega_n) = \omega_n V_0 - I > 0$, it follows that

$$\overline{\Pi}_0 = \frac{\mathbb{E}^{\overline{\mathbb{Q}}}\Pi_1(\omega)}{1+r} = \frac{\overline{q}(\omega_n(1+r)V_0 - I)}{1+r} \tag{25}$$

where $\overline{\mathbb{Q}} = (1-\overline{q}, 0, ..., 0, \overline{q})$ can be interpreted as a probability measure that assign a positive weight two states ω_1 and ω_n. By definition, for a given $\alpha \in (0,1)$, the α-maxmin criteria is equal to

$$\alpha\overline{\Pi}_0 + (1-\alpha)\underline{\Pi}_0 = \frac{\alpha\overline{q}(\omega_n(1+r)V_0 - I) + (1-\alpha)\underline{q}(\omega_{k+1}(1+r)V_0 - I)}{1+r} \tag{26}$$

which can also be written as

$$\alpha\frac{\mathbb{E}^{\overline{\mathbb{Q}}}\Pi_1(\omega)}{1+r} + (1-\alpha)\frac{\mathbb{E}^{\underline{\mathbb{Q}}}\Pi_1(\omega)}{1+r} = \alpha\frac{\overline{q}(\omega_n(1+r)V_0 - I)}{1+r} + (1-\alpha)\frac{\underline{q}(\omega_{k+1}(1+r)V_0 - I)}{1+r} \tag{27}$$

Let $\mathbb{Q}'^{\alpha} := \alpha\overline{\mathbb{Q}'} + (1-\alpha)\underline{\mathbb{Q}'}$ so that the resulting probability measure, written as a row vector, is equal to

$$\mathbb{Q}^{\alpha} = (\underbrace{\alpha(1-\overline{q})}_{\mathbb{Q}^{\alpha}(\{\omega_1\})}, 0, ..., 0, \underbrace{(1-\alpha)(1-\underline{q})}_{\mathbb{Q}^{\alpha}(\{\omega_k\})}, \underbrace{(1-\alpha)\underline{q}}_{\mathbb{Q}^{\alpha}(\{\omega_{k+1}\})}, 0, ..., 0, \underbrace{\alpha\overline{q}}_{\mathbb{Q}^{\alpha}(\{\omega_n\})}) \tag{28}$$

and note that this probability measure only assign positive a weight to the four following states, $\omega_1, \omega_k, \omega_{k+1}, \omega_n$. By definition,

$$\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}\Pi_1(\omega)}{1+r} = \frac{(\alpha(1-\overline{q})\Pi_1(\omega_1) + (1-\alpha)(1-\underline{q})\Pi_1(\omega_k) + (1-\alpha)\underline{q}\Pi_1(\omega_{k+1}) + \alpha\overline{q}\Pi_1(\omega_n)}{1+r} \tag{29}$$

Since $\Pi_1(\omega_1) = 0$ and $\Pi_1(\omega_k) = 0$ while $\Pi_1(\omega_{k+1}) = \omega_{k+1}(1+r)V_0 - I > 0$ and $\Pi_1(\omega_n) = \omega_n(1+r)V_0 - I > 0$, we obtain

$$\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}\Pi_1(\omega)}{1+r} = \frac{\alpha\overline{q}(\omega_n(1+r)V_0 - I) + (1-\alpha)\underline{q}(\omega_{k+1}(1+r)V_0 - I)}{1+r} \tag{30}$$

which is equation $(26)\square$

Proof of proposition 3. From equation (19), $V_0 = \frac{\underline{q}\omega_{k+1}(1+r)V_0+(1-\underline{q})\omega_k(1+r)V_0}{1+r}$ can be written as $\frac{\mathbb{E}^{\underline{\mathbb{Q}}}V_1(\omega)}{1+r} = V_0$, i.e., $\underline{\mathbb{Q}}$ can be thought of as a martingale measure. In the same way, $V_0 = \frac{\overline{q}\omega_n(1+r)V_0+(1-\overline{q})\omega_1(1+r)V_0}{1+r}$ can be written as $\frac{\mathbb{E}^{\overline{\mathbb{Q}}}V_1(\omega)}{1+r} = V_0$. It is easy to show that $\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}(V_1(\omega))}{1+r} = \alpha V_0 + (1-\alpha)V_0 = V_0$. Since the value of investing in period 0 is equal to $P_0 - I + \underbrace{\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}(V_1(\omega))}{1+r}}_{=V_0}$ while

the value of investing in period 1 is equal to $\frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}(\Pi_1(\omega))}{1+r}$, it follows it is optimal to invest in period 0 if $V_0 - I > \frac{\mathbb{E}^{\mathbb{Q}^{\alpha}}(\Pi_1(\omega))}{1+r} - P_0$. Note that this result does not depend upon the characteristics of the decision-maker and thus is true whether the decision-maker is risk-averse or not $\square$

References

[Braouezec and Grunspan, 2016] Braouezec, Y. and Grunspan, C. (2016). A new elementary geometric approach to option pricing bounds in discrete time models. *European Journal of Operational Research*, 249(1):270–280.

[Cerreia-Vioglio et al., 2015] Cerreia-Vioglio, S., Maccheroni, F., and Marinacci, M. (2015). Put–call parity and market frictions. *Journal of Economic Theory*, 157:730–762.

[Chateauneuf et al., 2007] Chateauneuf, A., Eichberger, J., and Grant, S. (2007). Choice under uncertainty with the best and worst in mind: Neo-additive capacities. *Journal of Economic Theory*, 137(1):538–567.

[Dixit et al., 1994] Dixit, A. K., Dixit, R. K., Pindyck, R. S., and Pindyck, R. (1994). *Investment under uncertainty*. Princeton university press.

[Elliott and Kopp, 2004] Elliott, R. and Kopp, P. (2004). Mathematics of financial markets, 2004.

[Etner et al., 2012] Etner, J., Jeleva, M., and Tallon, J.-M. (2012). Decision theory under ambiguity. *Journal of Economic Surveys*, 26(2):234–270.

[Gao and Driouchi, 2013] Gao, Y. and Driouchi, T. (2013). Incorporating knightian uncertainty into real options analysis: Using multiple-priors in the case of rail transit investment. *Transportation Research Part B: Methodological*, 55:23–40.

[Ghirardato et al., 2004] Ghirardato, P., Maccheroni, F., and Marinacci, M. (2004). Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, 118(2):133–173.

[Gilboa et al., 2008] Gilboa, I., Postlewaite, A. W., and Schmeidler, D. (2008). Probability and uncertainty in economic modeling. *Journal of Economic Perspectives*, 22(3):173–88.

[Harrison and Kreps, 1979] Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic theory*, 20(3):381–408.

[Musiela and Zariphopoulou, 2004] Musiela, M. and Zariphopoulou, T. (2004). A valuation algorithm for indifference prices in incomplete markets. *Finance and Stochastics*, 8(3):399–414.

[Nishimura and Ozaki, 2007] Nishimura, K. G. and Ozaki, H. (2007). Irreversible investment and knightian uncertainty. *Journal of Economic Theory*, 136(1):668–694.

[Pliska, 1997] Pliska, S. (1997). *Introduction to mathematical finance*. Blackwell publishers Oxford.

[Schröder, 2011] Schröder, D. (2011). Investment under ambiguity with the best and worst in mind. *Mathematics and Financial Economics*, 4(2):107–133.

[Staum, 2007] Staum, J. (2007). Incomplete markets. *Handbooks in operations research and management science*, 15:511–563.

[Taqqu and Willinger, 1987] Taqqu, M. S. and Willinger, W. (1987). The analysis of finite security markets using martingales. *Advances in Applied Probability*, 19(1):1–25.

[Varian, 1987] Varian, H. R. (1987). The arbitrage principle in financial economics. *Journal of Economic Perspectives*, 1(2):55–72.