On the Limiting Deterministic Case in McDonald-Siegel Real Option Model

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Abstract

In this note, we study the deterministic limit of McDonald-Siegel (1985) continuous time real option model in which the firm has the option to shut down the production at each time. We show that if the firm has already the option to wait, in the limiting deterministic case, the option to shut down the production becomes worthless because it is never used.

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Introduction

In a well known paper, Mc-Donald and Siegel (1985) have shown that when a firm has the possibility to temporarily and costlessly shut down the production, the present value of an investment project can be computed as a continuum of european call options, each one is given by the Black-Scholes formulae. However, no analytical formulae was provided. In their textbook, Dixit-Pyndick (1994) present an analytical formulae by using a “PDE approach”. Working with the limiting deterministic case as in Kongsted (1996), we shall here compute directly and analytically the value of an investment project when the firm has the option to shut down the production, and the option to wait.

This paper is organized as follows. In the first part, we present the basic set-up. In the second part, we compute the net present value (NPV) of the project when the firm has the option to temporarily shut down the production, i.e., we compute the value of a *continuum* of european call options in the limiting deterministic case. Finally, we compute the NPV of the investment project when the firm has the option to wait. As a result, in the limiting deterministic case, and only in this case, when the firm has the option to wait, adding the option to temporarily shut down the production has no marginal value[1].

The basic set-up

Let $(\Omega, \Im_t, \Im, Q)$ be our underlying filtered probability space, where $\Im_t$ is the natural filtration of the Brownian motion W_t and Q is the (unique) risk-neutral measure[2]. Let P_t be the price of some good (e.g., a commodity) traded on some market in continuous time. Under the risk-neutral measure Q, the price P_t of the good is assumed to follow a log-normal stochastic process:

$$P_t = P_0 \exp\left(\left(\alpha - 0.5\sigma^2\right)t + \sigma W_t\right) \Leftrightarrow \frac{dP_t}{P_t} = \alpha\, dt + \sigma\, dW_t$$

where W_t is the standard brownian motion, and $\alpha = r - \delta > 0$, where the parameter α is often called a (marginal) convenience yield (see e.g., McDonald-Siegel (1985)). As usual, we assume that the stochastic process $P = (P_t)_{t>0}$ is $\Im_t$-adapted (i.e., $\Im_t$-measurable for all *t*),

[1] In this paper, we only consider investment policy assuming that it is all-equity financed.
[2] We thus assume that the market is arbitrage-free and complete.

which means that at each time *t*, conditional on $\Im_t$, the sample path of the price is *known* between time 0 and *t* (included).

Suppose now that the firm has invested at time *t=0* in some tangible asset which allows to produce one unit of a good at each time $t \in [0,T]$, where T is the life time of the tangible asset, e.g., 10Y or 20Y. After *T*, the tangible asset is not anymore operational and has thus no scrapping value. To produce one unit of the good, it costs $\overline{C}$ to the firm. Since the firm sells its good at the market price P_t, the realized profit at a given time *t* is equal to:

$$P_t - \overline{C}$$

and may of course be positive or negative. The present value of the project, seen from time *t=0,* is thus equal to[3] the expected discounted flow of cash-flow under the risk-neutral measure:

$$V_0(T) = E^Q\left(\int_0^T \exp(-r\,t)\,(P_t - \overline{C})\,dt \middle| \Im_0\right) = \frac{P_0}{\delta}\left(1-\exp(-\delta T)\right) + \frac{\overline{C}}{r}\left(\exp(-r\,T)-1\right) \qquad (1)$$

Note importantly that $V_0(T)$ is *not* a function of σ, so that the formulae given in equation (1) also holds in the deterministic limit in which σ goes to zero. When the maturity goes to infinity, the present value reduces to:

$$\lim_{T\to\infty} V_0(T) = \frac{P_0}{\delta} - \frac{\overline{C}}{r}$$

Let I be the cost of the investment. When we consider the investment decision at time *t=0* only, it depends on the sign of the net present value (NPV) given below:

$$NPV_0(T) = \frac{P_0}{\delta}\left(1-\exp(-\delta T)\right) + \frac{\overline{C}}{r}\left(\exp(-r\,T)-1\right) - I \qquad \forall \sigma \geq 0 \qquad (2)$$

Optimal investment decision when the firm no option

- Invest in the project at time *t=0* if $NPV_0(T) > 0$
- Reject it *0* if $NPV_0(T) \leq 0$

[3] To compute this expectation, permute expectation and integration. Using the fact that W_t follows a normal distribution with mean zero and variance *t*., the problem reduces then to the computation of the expectation of an exponential of a Gaussian variable, which is easily computed using its moment generating function. Once this is done, it remains to compute a simple integral over time.

In the rest of the paper, we shall consider the case in which the firm has the option to shut down the production, as in McDonald and Siegel (1985), but also when it has the option to wait. These options are usually called "real options", because the underlying is not a financial assets, such as the value of a stock, but a real asset such as the price of a commodity.

Valuation of the option to shut down

When the firm has now the option to temporarily and costlessly shut down the production, the profit at a given time t is equal to:

$$\max\{0; P_t - \overline{C}\}$$

and this is in fact the payoff of a european call option of maturity t, of underlying P_t and strike $\overline{C}$. Its value, seen from time $t=0$, denoted $F(P_0,\overline{C},t,\delta,\sigma)$, is given by the Black-Scholes formulae[4].

$$F(P_0,\overline{C},t,\delta,\sigma) = P_0 \exp(-\delta t)\phi(d_1) - \exp(-rt)\overline{C}\,\phi(d_2)$$

with, as usual:

$$d_1 = \frac{\ln\left(P_0/\overline{C}\right) + \left(r - \delta + 0.5\sigma^2\right)t}{\sqrt{t}\,\sigma} \quad \text{and} \quad d_2 = \frac{\ln\left(P_0/\overline{C}\right) + \left(r - \delta - 0.5\sigma^2\right)t}{\sqrt{t}\,\sigma},$$ where $\phi(x)$ is distribution function of the standard gaussian random variable X. The present value of the project is thus equal to a *continuum* of european call options of maturity $t \in [0,T]$:

$$V_0(T,\sigma) = \int_0^T F\left(P_0,\overline{C},t,\delta,\sigma\right)dt$$

In general, computing this integral is difficult since there is no analytical formulae for $\phi(x)$. However, in the limiting deterministic case, i.e., when the volatility σ goes to zero, one can obtain an analytical formulae. It is easy to show that:

$$\lim_{\sigma\to 0}\phi(d_i) = 0 \;\; if \;\; P_0 \le \overline{C}\exp(-(r-\delta)t) \;\; \forall i = 1,2$$

$$\lim_{\sigma\to 0}\phi(d_i) = 1 \;\; if \;\; P_0 > \overline{C}\exp(-(r-\delta)t) \;\; \forall i = 1,2$$

[4] This is in fact the Black-Scholes formulae with continuous dividends δ.

Thus, seen from time $t=0$, the value of an european call option of maturity t is:

$$\lim_{\sigma \to 0} F(P_0, \overline{C}, t, \delta, \sigma) = \begin{cases} P_0 \exp(-\delta t) - \exp(-r\,t)\overline{C} & if \quad t > t^* \\ 0 & if \quad t \le t^* \end{cases} \tag{3}$$

where

$$t^* = \frac{\ln\left(\overline{C} \big/ P_0 \right)}{r - \delta} \tag{4}$$

We obviously consider the (interesting) case in which $P_0 < \overline{C}$ so that $t^* > 0$. Using now equation (3), we can compute the present value $V_0(T, \sigma)$ of the project when σ goes to zero. It reduces indeed to the following simple integral:

$$\begin{aligned} \lim_{\sigma \to 0} V_0(T, \sigma) &= \int_{t^*}^{T} \left(P_0 \exp(-\delta t) - \exp(-r\,t)\overline{C} \right) dt \\ &= \frac{P_0}{\delta} \left(\exp(-\delta t^*) - \exp(-\delta T) \right) + \frac{\overline{C}}{r} \left(\exp(-r\,T) - \exp(-r\,t^*) \right) \end{aligned}$$

Note that when $T \to \infty$, the above formulae reduces to:

$$\lim_{\substack{T \to \infty \\ \sigma \to 0}} V_0(T, \sigma) = \frac{P_0}{\delta} \left(\exp(-\delta t^*) \right) - \frac{\overline{C}}{r} \left(\exp(-r\,t^*) \right)$$

In the limiting deterministic case, the net present value of the project seen from time $t=0$ is equal to:

$$\lim_{\sigma \to 0} NPV_0(T, \sigma) = NPV_0(T) = \frac{P_0}{\delta} \left(\exp(-\delta t^*) - \exp(-\delta T) \right) + \frac{\overline{C}}{r} \left(\exp(-r\,T) - \exp(-r\,t^*) \right) - I$$

Optimal investment decision when the firm has the option to shut down the production

- Invest in the project at time $t=0$ if $NPV_0(T) > 0$
- Reject the project at time $t=0$ if $NPV_0(T) \le 0$

If the firm decides to invest, it will have to pay the investment cost *I* at time *t=0* , while the production will only start at time t^*, i.e., during the time interval $\left[0, t^*\right]$, the firm won't produce and will thus use its option to shut down the production.

Clearly, compare to the case in which the firm has not the option to shut down the production (see equation (1)), having this option adds (in general) positive value. It is easy to find examples in which the project will be rejected when the firm does not have the option to shut down the production, while it will be accepted when the firm has this option.

Valuation of the option to wait

We shall now determine the optimal investment policy when the firm has the option to wait. Let $\lim_{\sigma \to 0} NPV_0(t,T,\sigma) = NPV_0(t,T)$ denote the net present value of the project (seen from time *t=0*) when it is undertaken at time *t>0*. It is basically equation (2) except that now, the project starts at time *t* and not at time *t=0*. Recall that equation (2) is valid for all value of the volatility σ, so that it holds in the deterministic limit. It thus follows that:

$$\begin{aligned} NPV_0(t,T) &= \int_t^T \left(P_0 \exp(-\delta s) - \exp(-r s)\overline{C}\right) ds - \exp(-r t) I \\ &= \frac{P_0}{\delta}\left(\exp(-\delta t) - \exp(-\delta T)\right) + \frac{\overline{C}}{r}\left(\exp(-r T) - \exp(-r t)\right) - \exp(-r t) I \end{aligned}$$

Obviously, if the firm decides to undertake the project at time *t>0*, the investment cost *I* is paid at time *t* and is thus discounted at the risk free rate *r*. To find the optimal investment time, we thus need to solve the following optimization program[5]:

$$\max_{0 \le t \le T} NPV_0(t,T) = \frac{P_0}{\delta}\left(\exp(-\delta t) - \exp(-\delta T)\right) + \frac{\overline{C}}{r}\left(\exp(-r T) - \exp(-r t)\right) - \exp(-r t) I$$

It is easy to find that $\dfrac{dNPV_0(t,T)}{dt} = 0$ is equivalent to:

[5] Note the simplicity of the mathematical problem. In the stochastic case, the mathematical problem is much more complicated since one have to find the optimal exercising boundary which is a function of time. Numerical methods are thus necessary.

$$t^{**} = \frac{\ln\left(\overline{C} + r\,I \big/ P_0 \right)}{r-\delta} > 0 \qquad (4)$$

Since $\frac{d^2 NPV_0(t,T)}{dt^2} < 0 \ \forall t \in [0,T]$, the function $NPV_0(t,T)$ is strictly concave so that t^{**} is a global maximum. When the firm has the option to wait, it will invest at time t^{**} if the net present value seen from time *t=0* is positive[6], and will reject it otherwise.

Optimal investment decision when the firm has the option to wait

- Invest in the project at time t^{**} if $NPV_0(t^{**},T) > 0$
- Reject the project at time *t=0* if $NPV_0(t^{**},T) \leq 0$

Note that $t^{**} > t^{*}$, which implies that if we add the option to shut down the production to the option to wait, its *marginal value is zero* since it is never used by the firm. Put it differently, in the limiting deterministic case, if the firm had to choose between the option to shut down and the option to wait, it will choose the option to wait.

When the firm has the both the option to shut down the production and the option to wait, in the limiting deterministic case, and only in this case, the option to shut down the production becomes *worthless* since the firm does not use it.

To better understand this simple result, assume for simplicity that the convenience yield is zero, i.e., $\delta = 0$.

- When the firm has the option to shut down the production and has invested in the project, it will start the production as soon as P_t is higher than $\overline{C}$, i.e., at time $t^{*} = \frac{\ln\left(\overline{C} \big/ P_0 \right)}{r}$.

[6] Note that if the net present value is positive seen from time *t*, it is also positive seen from time t^{**}.

- When the firm has the option to wait, it will not invest as soon as the price becomes higher than the cost, but as soon as the price becomes *sufficiently higher* than the cost, i.e., at time $t^{**} = \frac{\ln\left(\overline{C} + r\,I \middle/ P_0 \right)}{r}$.

As we have seen, since $t^{**} > t^{*}$, when the firm already has the option to wait, the marginal value of the option to shut down the production is zero. This result is however not true in the stochastic case, since the probability that the price becomes lower than the cost (when the firm has invested) is positive so that the option to shut down the production may be used in the future, and thus is not worthless. On the contrary, in our limiting deterministic case, since the price dynamics becomes a simple exponential function of the time, there is no chance that it becomes lower than the cost. As a consequence, the option to shut down the production becomes worthless when the firm already has the option to wait. Of course, this result crucially depends of the deterministic limit of the log-normal stochastic process. It might not be true for the deterministic limit of another stochastic process.

Conclusion

In this note, we have provided a simple natural way to determine the optimal investment decision in which both finite[7] and infinite maturity cases can be treated analytically. In the second case, as in Dixit-Pyndick (1994), the parameter δ has to be positive to insure that the value of a continuum of European options remains finite.

References

Dixit Avinash, and Robert Pindyck, 1994, *Investment under Uncertainty*, Princeton University Press.

Kongsted Hans, 1996, "Entry and exit decisions under uncertainty: The limiting deterministic case", *Economics Letters*, pp. 77-82

[7] In the stochastic case, the problem can be solved analytically only when the maturity is infinite.

Pindyck Robert, 1991, "Irreversibility, Uncertainty and Investment", *Journal of Economic Literature*, September, pp. 1110-1148.

McDonald Robert, and Siegel Daniel 1985, "Investment and the Valuation of Firms When There is an Option to Shut Down", *International Economic Review*, pp. 331-349.