Translating the Greek: The Real Meaning of Call Option Derivatives

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The derivatives of a call option's price with respect to the stock price, volatility, risk-free rate, exercise price and time to expiration indicate how the option price will change as the underlying inputs change. Option users must understand the significance of these derivatives. Some of the traditional interpretations of these concepts are misleading. At best, they provide only partial explanations for what is really happening.

Decomposing a call into a margin transaction and an insurance policy provides clearer interpretations. For example, increased volatility has no impact on the value of the margin-position component of the call. It does, however, increase the insurance value of the option. A longer time to expiration (traditionally viewed as beneficial, giving the asset more time to appreciate) in fact increases the value of the margin position but can increase or decrease the insurance value.

For ordinary European calls on assets that have no underlying payouts or convenience yields, five variables affect price—the underlying asset price, the volatility (standard deviation of returns), the risk-free rate, the option's exercise price and the time to expiration. Understanding how a change in any of these variables affects a call option's price is important for the investor or speculator using call options.

To quantify the dynamics, one must first take the partial derivative of the call price with respect to each variable. These results, usually referred to as the *comparative statics*, are easily obtained if the option has a known, closed-form formula (in this case, the celebrated Black-Scholes model).[1] One must then apply common intuition to see how a change in one variable changes the call price.

Unfortunately, not all the results are particularly intuitive. For example, how does the risk-free rate affect the call option price? A call is like a leveraged transaction. An investor can take a leveraged position in an asset by buying it on margin or by buying a call on the asset. One school of thought has it that, the higher the risk-free rate, the less attractive the margin trade is. The funds saved by buying the option instead of a fully margined position in the asset (i.e., no borrowing) can earn more interest. It has also been argued that, because the lower bound on a European call must rise when the risk-free rate increases, call price should also rise.

These explanations are only partially correct. For a better understanding of the effect of interest rates on call option prices, it is helpful to break down a call option into its two underlying components—a margin transaction and an insurance policy. By doing so, one can obtain better insights not only into the effect of interest rates on call prices, but also into the effects of the other four variables.

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DISSECTING A CALL OPTION TRANSACTION

The basic intuition here is simple and has appeared many times before.[2] A European call option is a leveraged position in an asset plus an insurance policy. The leverage transaction entails borrowing the present value of X dollars (Xe^{-rT}) and purchasing one unit of the asset, which sells for S. Let us refer to this transaction as the *margin transaction* and denote its value as:

$$m = S - Xe^{-rT}.$$

We can think of m as the amount of money the investor must put up.

Now suppose the investor purchases an insurance policy on the stock. This is nothing more than a European put option allowing the holder of the asset to sell it for X dollars after the passage of time T. This policy requires the up-front payment of a premium, p. Since most individuals are familiar with insurance, they should find it easier to understand a put than a call. In fact, a put should probably be recognized as the more basic instrument, out of which a call can be created.

The value of the complete transaction needed to replicate a call is the sum of the values of the margin position and the insurance policy:

$$c = m + p = S - Xe^{-rT} + p.$$

This result should be easily recognized as the familiar put-call parity. Note that the loan is truly risk-free, because the stock insured with the put is sufficient to guarantee that X dollars will be available at maturity.

If the asset is a common stock, an actual margin transaction has one advantage over a call—namely, it confers the right to vote. However, the call has an important advantage over the margin transaction. The call is not subject to Federal Reserve/NASD/stock exchange margin requirements. Even though the call premium must be paid in full, the call effectively permits borrowing more than 100% of the asset's value; thus even a portion of the insurance premium can be borrowed. This is what happens when an investor buys an out-of-the-money call.[3]

Traditional Interpretations

The Black-Scholes formula is given as:

$$c = SN(d_1) - Xe^{-rT}N(d_2)$$

where

$$d_1 = \frac{ln(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and

$$N(.) = \textit{cumulative normal probability}.$$

Differentiating the Black-Scholes model with respect to each variable provides the comparative statics, which are presented in Table 1.[4] Note that these are partial derivatives, meaning that while we allow one variable to change, we hold all other variables constant. The critical nature of this assumption will be demonstrated later.

Now consider the traditional interpretation of these results. Equation C1 in the table implies that, *the higher the asset price, the more valuable the call.* This effect, referred to as the option's *delta*, is obtained because the call is an option to buy the asset at a fixed price. If the asset increases in value, the call must logically be more attractive, hence carry a higher price.

Table 1. Derivatives of a Call Option

(C1) *Asset Price*: $\frac{\partial c}{\partial S} = N(d_1) > 0$ (*Delta*)

(C2) *Exercise Price*: $\frac{\partial c}{\partial X} = -e^{-rT}N(d_2) < 0$

(C3) *Risk-Free Rate*: $\frac{\partial c}{\partial r} = TXe^{-rT}N(d_2) > 0$ (*Rho*)

(C4) *Volatility*: $\frac{\partial c}{\partial \sigma} = S\sqrt{T}N'(d_1) > 0$ (*Vega*)

(C5) *Time to Expiration*: $\frac{\partial c}{\partial T} = \frac{S\sigma}{2\sqrt{T}}N'(d_1) + rXe^{-rT}N(d_2) > 0$
(*− 1 times Theta*)

Note: The value $N'(d_1)$ is the derivative of the standard normal distribution function with respect to d_1.

Equation C2 tells us that, *the higher the exercise price, the less valuable the call.*[5] For the call to be valuable at expiration, the asset price must exceed the exercise price. Thus a higher exercise price presents a more difficult hurdle for the asset to scale. Alternatively, holders of a call would find it less attractive if they had to pay more to buy the asset at expiration.

Equation C3 says, *the higher the risk-free rate, the more valuable the call.*[6] This derivative, referred to as the options's *rho*, is difficult to interpret. An increase in interest rates makes a margin trade more expensive and leaves the call the more attractive means of establishing a leveraged position. In addition, a rise in interest rates raises the minimum value of the call. However, as I shall demonstrate later, these explanations give only part of the picture.

Equation C4 implies that, *the higher the volatility, the more valuable the call.*[7] This effect, referred to as the option's *vega*, seems the most obvious. As volatility increases, the potential gain from a call expiring in the money is much greater. The fact that the upside potential is coupled with greater downside risk is of no consequence, because the downside loss is limited. This seems to indicate that volatility has value because of its effect on upside movements; as I shall show later, this is not exactly what is happening.

Equation C5 says that, *the longer the time to expiration, the more valuable the call.* This derivative, which equals −1 times the option's *theta*, means that a call with a longer time to expiration affords more time for the underlying asset's price to move in a favorable direction. Once again, this interpretation gives only one side of the picture.

ALTERNATIVE INTERPRETATIONS

Table 2 presents the derivatives of the call option price when the call option is broken down into its two components—a margin transaction and an insurance policy. To facilitate the interpretation, consider an investor who is contemplating a leveraged position in the asset combined with an insurance policy that sets the lowest effective value of the asset at the face value of the loan. This position obviously replicates a call. The investor can choose any amount of debt and insurance and any length for the insurance policy and the loan. In addition, over the next instant in time, the asset's price, its volatility or the risk-free rate could change, making the position more or less expensive.

Table 2. Derivatives of the Margin and Insurance Positions

Margin Position

(M1) *Asset Price*: $\frac{\partial m}{\partial S} = 1 > 0$

(M2) *Exercise Price*: $\frac{\partial m}{\partial X} = -e^{-rT} < 0$

(M3) *Risk-Free Rate*: $\frac{\partial m}{\partial r} = TXe^{-rT} > 0$

(M4) *Volatility*: $\frac{\partial m}{\partial \sigma} = 0$

(M5) *Time to Expiration*: $\frac{\partial m}{\partial T} = rXe^{-rT} > 0$

Insurance Position

(I1) *Asset Price*: $\frac{\partial p}{\partial S} = N(d_1) - 1 < 0$

(I2) *Exercise Price*: $\frac{\partial p}{\partial X} = e^{-rT}[1 - N(d_2)] > 0$

(I3) *Risk-Free Rate*: $\frac{\partial p}{\partial r} = -TXe^{-rT}[1 - N(d_2)] < 0$

(I4) *Volatility*: $\frac{\partial p}{\partial \sigma} = S\sqrt{T}N'(d_1) > 0$

(I5) *Time to Expiration*: $\frac{\partial p}{\partial T} = \frac{S\sigma}{2\sqrt{T}}N'(d_1) - rXe^{-rT}[1 - N(d_2)] \gtrless 0$

Recall that the margin transaction involves borrowing the present value of X dollars by issuing a promissory note discounted at the risk-free rate r and promising to pay back X dollars after the end of a period of time T. The difference between the asset's price, S, and the present value of the exercise price, Xe^{-rT}, is the amount of money the investor will have to put up to establish the trade. This can be thought of as the margin.[8] The insurance policy is obtained by paying a premium, p. If the asset is worth less than X dollars on the expiration day, the policy pays the investor X dollars for the asset.

The Black-Scholes model does not permit default, so we must assume the insurer is solvent when the policy expires.[9] The Black-Scholes model for pricing a put shows that the put premium is worth

$$Xe^{-rT}[1 - N(d_2)] - S[1 - N(d_1)],$$

where $N(d_1)$ is the call option's delta. This means that the party holding the put (the insured party) effectively holds a long position in $1 - N(d_2)$ units of a risk-free bond worth Xe^{-rT} today and a short position in $1 - N(d_1)$ units of the asset. Because T, S, $N(d_1)$ and $N(d_2)$ are constantly changing, the investor will need to adjust the proportions of the risk-free bond and the short position in the asset. This, of course, is the process of *dynamic hedging*, the basis for many option strategies, including portfolio insurance.

The insurer has a short position in the put. To guarantee that the insurer will pay off, the Black-Scholes model implicitly assumes that the insurer will create a synthetic long position in the put. One way to do this is to hold $1 - N(d_2)$ units of the risk-free bond and sell short $1 - N(d_1)$ units of the asset, adjusting this position as the values change. Insurer solvency is thus guaranteed by having the insurer dynamically replicate the policy liability with a long position in bonds and a short position in the asset.

It is also useful to think about the position of a call writer. An uncovered call writer is not only lending the funds to establish the leveraged position, but also shorting the asset and writing the insurance policy that covers the investor's losses beyond the exercise price. A covered call writer is only lending the funds and writing the insurance.

Effect of a Higher Asset Price

Equation M1 in Table 2 indicates that the higher the asset price, the more the investor must put up to establish the margin position. In other words, with the face value of the loan held constant, the more expensive the asset, the more personal funds the investor will have to lay out. For every dollar increase in the asset price, the investor must put up an additional dollar.

Equation I1 in Table 2 indicates that the higher the asset price, the less expensive the insurance. This makes sense because the insurer faces less risk if the asset is worth more, so it can charge a lower premium. Recall that, to guarantee solvency, the insurer is short $1 - N(d_1)$ units of the asset. Thus it reduces the premium by $1 - N(d_1)$ per unit increase in the price of the asset, which is consistent with the derivative of $N(d_1) - 1$.

The combined effects of the two derivatives produce the derivative in Equation C1 in Table 1. The insurance is cheaper, but this is more than offset by the fact that the margin position will cost more. Thus a higher asset price raises the value of a call by increasing the amount the investor must pay to establish the margin position more than it reduces the cost of insuring against downside risk.

Effect of a Higher Exercise Price

For the margin transaction, a higher exercise price means the investor is using more leverage or less margin. From Equation M2 in Table 2 we see that, the more money the investor borrows (the higher X), the less of his own money he will require. For every additional dollar the investor promises to pay back at expiration, the initial outlay is reduced by the present value of that dollar; hence the derivative is $-e^{-rT}$.

If the investor chooses a higher exercise price, he is asking the insurer for greater coverage; the effect is similar to lowering the deductible. Clearly, the cost of the insurance will go up. In the risk-neutral world in which options can typically be valued, the probability that a call will be exercised is $N(d_2)$; thus $1 - N(d_2)$ is the risk-neutral probability that the put will be exercised or, in other words, the risk-neutral probability of an insurance claim.

Equation I2 in Table 2 is positive and has a simple interpretation. The insurance premium will go up by the present value of the additional coverage, e^{-rT}, per unit increase in the level of coverage times the risk-neutral probability of a claim, $1 - N(d_2)$. In other words, the insurer will need to set aside additional funds in risk-free bonds in the amount of the present value of the increment to the exercise price times the risk-neutral probability that a claim will be made.

The combined effects of the less expensive margin position and the more expensive insurance make up Equation C2 of Table 1. Because $1 - N(d_2)$ is a probability and must, by definition, be no greater than unity, the margin effect dominates. The overall derivative is interpreted as the incremental reduction in the margin cost, e^{-rT}, times the risk-neutral probability that an insurance claim will not be made, $N(d_2)$. The higher the probability of a claim, the more the insurer will increase the premium, thus reducing the overall effect.

Effect of a Higher Risk-Free Rate

What happens if the risk-free rate increases the instant before the position is established? If the lender must charge a higher interest rate, the investor must put up more personal funds to establish the margin position. Equation M3 in Table 2 shows that the additional funds required are TXe^{-rT}. This expression is the incremental interest the investor must pay.[10]

If the risk-free rate goes up, Equation I3 in Table 2 tells us that the insurance premium goes down. Recall that, in order to ensure solvency, it is assumed that the insurer sets aside sufficient funds to pay off the claim. It must invest the present value of X dollars times the probability of a claim in risk-free bonds. We can think of this as escrowing the expected liability.

With a higher interest rate, the insurer can set aside a smaller amount of funds. Specifically, for a given increase in r, it can reduce the escrowed funds by the additional interest it would earn, which is TXe^{-rT}, times the risk-neutral probability of a claim, $1 - N(d_2)$. It can thus reduce the premium by this amount.

The overall effect of an interest rate increase is to raise the cost of establishing the margin position while lowering the cost of the insurance. Equation C3 in Table 1 shows that the margin effect dominates, and the overall derivative is the incremental interest, TXe^{-rT}, times the risk-neutral probability of no insurance claim, $N(d_2)$. From the investor's point of view, this term represents the additional cost of establishing the margin position times the risk-neutral probability that he will not receive any of the funds back through an insurance claim. In other words, the additional cost of the margin position is lost completely, but some of this cost could be recouped if an insurance claim were made.

The traditional interpretation that an increase in the interest rate raises the value of the call because it represents margin interest is seen to be partially correct. However, it must be tempered by the fact that the full additional interest is not reflected in an increase in the call price. What matters is the expected incremental outlay, which differs from the full incremental outlay because some of the outlay might be recouped if the asset price goes down and an insurance claim is made.

Effect of an Increase in Volatility

Equation M4 in Table 2 shows that volatility has no effect on the value of a margin position. There is no upside benefit or downside cost.

From Equation I4 in Table 2 we see that the insurer will charge a larger premium with an increase in volatility. The increase will equal the incremental amount the insurer expects to pay out at expiration, conditional on a claim being made. The appendix provides a proof of this.

When we combine these effects, we obtain Equation C4 in Table 1. Note, however, that our new interpretation is quite different from the traditional one. In the traditional view, higher volatility increases call value because it increases the upside potential without increasing the downside risk. Yet there is nothing in Equations M4 and I4 in Table 2 that implies upside gains. The benefit of greater volatility manifests itself strictly in its effect on the downside.

A call is a leveraged stock position with limited downside loss. Greater volatility increases the value of the limited downside risk. The tendency to think of a call in terms of its attractiveness in a rising market should therefore be tempered by recognition of its usefulness in a falling market.

Effect of a Longer Time to Expiration

Suppose the investor is considering establishing a position with a longer holding period. The loan will be paid back later, and the insurance will cover a longer term.

Equation M5 in Table 2 indicates that the effect of a longer loan is to increase the cost of establishing the margin position. In other words, if the investor borrows the same amount of money but promises to pay it back later, the loan will be more costly. The derivative is the incremental discounted interest for a change in T, rXe^{-rT}, which reflects the additional funds the investor will have to put up.

Equation I5 in Table 2 indicates that there are two effects on the insurance premium. The first term is a risk effect, reflecting the incremental amount the insurer expects to pay out at expiration, conditional on a claim being made. The longer the period of coverage, the greater the risk assumed by the insurer, hence the higher the premium. (The appendix provides a proof.)

The second effect is the incremental interest the insurer can earn. Remember that the insurer holds bonds worth the present value of X times the probability of a claim. With a longer period of coverage but the same ultimate potential payout, X, the insurer can deposit a smaller amount of funds, hence earn more interest. It can therefore reduce the premium charged to the insured by the incremental interest times the risk-neutral probability of a claim, $1 - N(d_2)$.

Thus insurance could be either more or less costly, given a longer time to expiration.[11] It would tend to be less costly, the higher the interest rate and more costly, the greater volatility. So the margin will be more costly and the insurance could be either more or less costly. The combined effect, as indicated in Equation C5 in Table 1, is that the overall insured, leveraged position is more costly.

The overall expression has two components. The first reflects the increased risk to the insurer. The second expression is the incremental interest times the risk-neutral probability of no insurance claim, $N(d_2)$. What this means is that the amount by which the insurer can reduce the premium, given the increased interest it can earn on the risk-free bonds, is more than offset by the additional interest the investor pays on the margin transaction. The effect of the additional interest is only a partial effect, however, reflecting the risk-neutral probability that there will be no insurance claim. In the event of a claim, the investor could recoup the full amount of the additional outlay; the risk-neutral probability is $N(d_2)$, however, that such a claim will not be made.

A CAVEAT ON PARTIAL DERIVATIVES

Partial derivatives indicate the change in a function (here the call price) for a given change in one variable, treating all other variables as if they did not change. Now is a good time to examine how important this assumption is.

Recall that an increase in volatility has no effect on the margin position but increases the cost of the insurance. An increase in volatility is, therefore, strictly a downside effect, reflecting the increased insurance premium. There is no upside effect, as the traditional interpretation would have it.

This interpretation is correct, but *only under the limiting assumption behind partial derivatives*. Consider a simple example using the binomial model of Cox, Ross and Rubinstein.[12] Suppose the asset price is 100 and can go to either 115 or 90 one period later. The up and down factors are thus $u = 1.15$ and $d = 0.9$. The risk-free rate is 5%, so $r = 1.05$.

We can easily price a call option with an exercise price of 100. First we find θ (the risk-neutral probability) as:

$$\theta = \frac{r - d}{u - d} = \frac{1.05 - 0.9}{1.15 - 0.9} = 0.6.$$

Then $1 - \theta = 0.4$. Let c^+ and c^- be the prices of the call when the asset goes up to 115 or down to 90, respectively. With $X = 100$, $c^+ = 15$ and $c^- = 0$. The price of the call is given as:

$$c = \frac{\theta c^+ + (1 - \theta)c^-}{r} = \frac{0.6(15) + 0.4(0)}{1.05} = 8.57.$$

If, as we stated earlier, the margin position is unaffected by increased volatility, we should be able to reduce the lower of the two possible stock prices, thereby increasing the insurer's potential liability, and obtain a higher call price. In other words, what if the asset could now go down to 80 instead of 90?

We have added some downside volatility, yet the call price goes up. Under the traditional interpretation, this presents a bit of a paradox. We cannot benefit from greater potential upside gains and we do not care whether the asset ends up at 90 or 80. Recalculating with $d = 0.8$, we would now find that $\theta = 0.7143$ and $c = 10.20$. How did we obtain this paradoxical result that an increase in only downside volatility still manages to increase the call price?

The key is that we held the asset price constant, thereby violating one of the basic principles of finance. If the asset was worth 100 when it could go up to 115 or down to 90, it cannot be worth 100 when it can now go up to 115 or down to 80 (holding the interest rate constant). How can we reconcile this result with the notion of rational investors?

Suppose we assume that investors have logarithmic utility, meaning that an asset worth 115 will provide log(115) = 4.74 utils and one worth 90 will provide log(90) = 4.50 utils.[13] The expected utility from owning the stock is

$$4.74q + 4.50(1 - q),$$

where q is the actual (not risk-neutral) probability of an up move. The *certainty-equivalent* asset price at expiration—the value an investor would accept for certain in lieu of the risky investment—is obtained by inserting the expected utility into the exponential function. This value is then discounted at the risk-free rate to obtain the current price of 100. We must thus find the actual probability of an up move.

The formula for q is:

$$q = \frac{\ln r - \ln d}{\ln u - \ln d}$$

Note how similar the actual probabilities are to the risk-neutral probabilities. Plugging $r = 1.05$, $u = 1.15$ and $d = 0.9$ into the above formula gives $q = 0.6289$ and $1 - q = 0.3711$. These probabilities would lead to a price of 100.

To recap the problem, we have an asset held by investors with logarithmic utility that can go up to either 115 or down to 90 while the risk-free rate is 5%. The asset is worth 100. We know that a call with an exercise price of 100 is worth 8.57. Now we change the lower stock price from 90 to 80. Plugging the values of $r = 1.05$, $u = 1.15$ and $d = 0.8$ into the formula for q gives the actual probabilities as $q = 0.7493$ and $1 - q = 0.2507$. Note that to keep the stock price at 100, the probability of an up move had to increase. The new values for the risk-neutral probabilities, θ and $1 - \theta$, are now 0.7143 and 0.2857, as noted above.

Thus we injected what appeared to be only downside volatility. Yet by holding the stock price constant, we were implicitly increasing the probability of an upward move, which resulted in the call price increasing.

What would have happened if the probabilities had not changed? Let us hold q and $1 - q$ at 0.6289 and 0.3711. Then S would be:

$$S = \frac{e^{0.6289 \ln(115) + 0.3711 \ln(80)}}{1.05} = 95.72.$$

With $S = 95.72$, we have $u = 115/95.72 = 1.20$, $d = 80/95.72 = 0.84$. Then $\theta = 0.5833$ and $1 - \theta = 0.4167$. The call is now worth:

$$c = \frac{0.5833(15) + 0.4167(0)}{1.05} = 8.33.$$

which is lower!

In other words, we injected an apparent increase in volatility, albeit downside, and obtained a lower call price. This makes intuitive sense, because we incorporated the effect that volatility has on the stock price.

The moral is that whenever one interprets partial derivatives, one should always remember that they represent the effect of a change in *one* variable on the function. In reality, a change in one variable is more than likely to interact with other variables. Under the assumptions of partial derivatives (that is, only one variable changes), the correct interpretation of, for example, an option's vega is that it increases the cost of the insurance and has no effect on the leverage. The truth is that volatility is likely to have an effect on leverage that swamps its other, more obvious, effect on insurance. A similar conclusion holds for changes in interest rates.[14]

APPENDIX

It is well known that $SN(d_1)$ is the expected stock price at expiration, conditional on the stock price ending up greater than the exercise price. The insurer's incremental cash flow per change in σ is the partial derivative of $-S[1 - N(d_1)]$ with respect to sigma. The expression to be differentiated is interpreted as -1 (because it is a liability) times the risk-neutralized expected value of the stock price, given that the stock price ends up below the exercise price.

Because S is unaffected by σ, we need only evaluate the simple expression $SN(d_1)$. Its derivative with respect to d_1 is $SN'(d_1)$, which is the change in the expected stock price, conditional on the stock price exeeding the exercise price, from changing the number of standard normal deviations (d_1). Now we multiply this by the number of incremental standard normal deviations resulting from a change in the instantaneous standard deviation, σ. We obtain this second value as follows.

The return on the stock over the remaining life of the option is $\Delta S/S$. If the stock follows the standard lognormal diffusion with time-homogenous parameters, the variance of $\Delta S/S$ can be found by stochastic integration over the remaining life. The result will be $\sigma^2 T$. Thus the standard deviation of $\Delta S/S$ is $\sigma \sqrt{T}$. Differentiating this expression with respect to σ gives $\sqrt{T}$, which is the number of incremental standard deviations of the return over the remaining life for a change in the instantaneous standard deviation.

Thus the product of $\sqrt{T}$ and $SN'(d_1)$ gives the expected incremental payout of the insurer conditional on an insurance claim being made.

Time to Expiration

To examine the effect of increasing the time to expiration, we once again interpret $SN'(d_1)$ as the change in the risk-neutralized expected stock price, conditional on the stock price exeeding the exercise price, from changing the number of standard normal deviations (d_1). The number of incremental standard normal deviations from changing T is found by differentiating (with respect to T) the expression given above for the standard deviation of the stock return over the remaining life of the option, which was $\sigma\sqrt{T}$. This gives $\sigma/2\sqrt{T}$. The product of these two expressions is the overall derivative and is interpreted as the expected incremental payout of the insurer conditional on an insurance claim being made.

FOOTNOTES

1. See F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, May-June 1973.
2. An excellent treatment is D. Galai, "Characterization of Options," *Journal of Banking and Finance*, December 1977.
3. Here I define at the money as $S = Xe^{-rT}$.
4. These results are already well known; however, taking these derivatives requires the application of Leibnitz's rule and some useful tricks. An excellent presentation of the mathematical details is found in T. E. Conine Jr. and M. Tamarkin, "A Pedagogic Note on the Derivation of the Comparative Statics of the Option Pricing Model," *The Financial Review*, November 1984.
5. It should be noted that with ordinary calls, the exercise price does not change. Thus the interpretation of a "higher exercise price" must be that an investor is considering two calls alike in all respects except exercise price. For options with a changing exercise price, see S. Fischer, "Call Option Pricing When the Exercise Price is Uncertain, and the Valuation of Index Bonds," *The Journal of Finance*, March 1978.
6. The assumptions underlying the Black-Scholes model dictate that the risk-free interest rate does not change. If the risk-free rate follows a lognormal diffusion process, a slight variation of the model is obtained. See R. C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, Spring 1973.
7. The Black-Scholes assumptions also require constant volatility. There have been many models that price options with changing volatility. See, e. g., J. Hull and A. White, "The Pricing of Options with Stochastic Volatility," *The Journal of Finance*, June 1987.
8. Often the margin is expressed as a percentage of the value of the asset. Thus, in this case, the margin percentage is $(S - Xe^{-rT})/S$.
9. For an option pricing model when the writer can default, see H. Johnson and R. Stulz, "The Pricing of Options with Default Risk," *The Journal of Finance*, June 1987.
10. Recall that in a discounted loan the interest is paid in advance. Thus the lender remits a lower amount and the investor must make up the difference by putting up additional personal funds to buy the asset.
11. This result is equivalent to the well-known fact that the theta of a European put can be either positive or negative.
12. J. C. Cox, S. A. Ross and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, September 1979.
13. For an excellent, concise treatment of utility functions, see M. Kritzman, "What Practitioners Need to Know About Utility," *Financial Analysts Journal*, May/June 1992.
14. I thank Gary Gastineau, Roger Clarke, Charles Smithson, Mike Hemler, Meir Schneller, Dean Leistikow, Don Rich and Bill Trainor for their helpful comments.