Technical Note

Statistical Models for Financial Volatility

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This paper develops tools for measuring and forecasting volatility when it varies over time. A variety of popular models for conditional variances, including ARCH, GARCH and EGARCH, are presented and compared. The results suggest that volatility is forecastable. Such predictions may affect portfolio decisions. Volatility may consequently be priced, leading to time-varying risk premiums.

Financial market **volatility** is predictable. This observation has important implications for asset pricing and portfolio management. Investors seeking to avoid risk, for example, may choose to adjust their portfolios by reducing their commitments to assets whose volatilities are predicted to increase or by using more sophisticated dynamic diversification approaches to hedge predicted volatility increases. In a market in which such strategies operate, equilibrium asset prices should respond to forecasts of volatility, as well as to the risk aversion of investors. This is particularly true of the markets for derivative assets such as options and swaps, where the volatility of the underlying asset has a profound effect on the value of the derivative.

As will become clear, a prediction of high volatility is really just a prediction of high variance—a prediction that the potential size of a price move is great. Thus even perfect predictability of variances does not mean perfect predictability of the size of market moves or of their direction. Volatility forecasting is a little like predicting whether it will rain: You can be correct in predicting the probability of rain, but still have no rain.

Volatility clustering is one of the oldest noted characteristics of financial data. It tells us something about the predictability of volatility. If large changes in financial markets tend to be followed by more large changes, in either direction, then volatility must be predictably high after large changes. This is, in fact, how traders typically predict volatility. They measure standard deviations over various periods and use what they judge to be the most appropriate moving average to predict volatility. Some adjust standard deviations to reflect recent events, recognizing that these may contain additional information useful in predicting volatility. Traders who deal in longer-lived assets, however, may believe that volatility in the distant future is insensitive to current information.

Traders price assets based in part on such volatility forecasts. It is possible that better volatility forecasts in either the short or long run could lead to better estimates of fundamental asset values. It remains to be seen whether the market already reflects the best available forecasts.

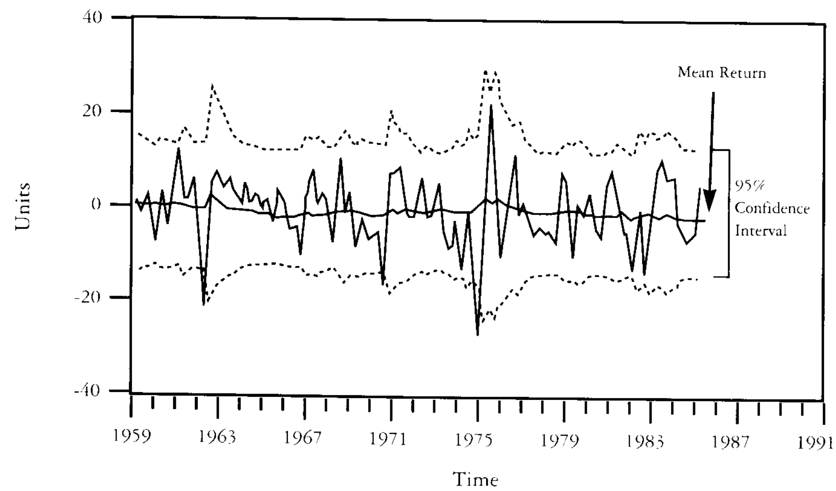
This paper discusses a systematic way to use statistics to find the best forecast of volatility. Using statistics, it is possible to determine whether recent information is more important than old information, and how fast information decays. It is possible to determine whether volatility is equally sensitive to market up moves and down moves, and whether it is proportional to the size of past returns or to the square of returns or to some other function. Forecasts can be made over short horizons or long horizons, and forecasts can be made for a single asset's volatility or for a whole set of asset variances and covariances.

Once the statistical properties of variance forecasting have been established, examining the implications for financial markets is straightforward. In particular, models of time-varying risk premiums, time-varying hedge ratios, time-varying betas and option pricing can be developed. This paper introduces many of the central concepts of volatility forecasting and then briefly discusses applications of the ideas to various financial market questions.

Models of Volatility

Figure A plots the quarterly returns for the S&P 500 over the 1959–84 period. Clearly, the largest positive and negative returns occurred in the mid-seventies. The dashed lines depict a time-varying 95% confidence interval drawn using a statistical approach to forecasting volatility one quarter in advance. Note that the interval is wider during the more volatile periods. The middle line shows the mean, which is proportional to the variance. While the mean moves very little compared with the *ex post* fluctuations in the series, notice that it does rise in the 1970s—the period of highest volatility—implying that risk-averse investors demand higher

Figure A S&P 500 Quarterly Returns*



* Based on estimated conditional variance and conditional mean using a GARCH-M model.

Glossary

▶***Conditional Mean:*** The expected value of a random variable when some other random variables are known. It is generally a function of these random variables.

▶***Conditional Variance:*** The variance of a random variable when some other random variables are known. It is generally a function of these random variables.

▶***Factor Loadings:*** Constants that measure the importance of each factor in explaining the returns on an asset.

▶***Heteroskedasticity:*** Unequal variances of a set of random variables.

▶***Volatility:*** A tendency for prices to fluctuate widely.

▶***Volatility Clustering:*** A tendency for large swings in prices to be followed by large swings of random direction.

risk premiums in periods of high volatility.

The Statistical Model

The original tool for analyzing volatility forecasts was the autoregressive conditional heteroskedasticity or ARCH model, introduced in Engle.[1] This was generalized to GARCH (generalized ARCH) by Bollerslev and has been further extended by Nelson's EGARCH (exponential GARCH) and by Schwert.[2] These are all models of conditional variance. In order to understand their similarities and differences, it is important to clarify the difference between conditional and unconditional moments.

Let y_t be the return on an asset received in period t. Let E represent mathematical expectation. Then the mean of the return can be called μ, and

Eq. 1

$$Ey_t = \mu.$$

This is the unconditional mean, which is not a random variable. The **conditional mean**, m_t, uses information from the previous period and can generally forecast more accurately. It is given by:

Eq. 2

$$m_t \equiv E[y_t|F_{t-1}] \equiv E_{t-1}[y_t].$$

This is in general a random variable depending on the information set F_{t-1}. Note that, although $y - \mu$ can be forecast, $y_t - m_t = \varepsilon_t$ cannot, using the information in F_{t-1} alone.

The unconditional and conditional variances can be defined, respectively as:

Eq. 3

$$\sigma^2 \equiv E[y_t - \mu]^2 = E[y_t - m_t]^2 + E[m_t - \mu]^2$$

Eq. 4

$$h_t \equiv E_{t-1}[y_t - m_t]^2.$$

The **conditional variance** potentially depends upon the information set.

Volatility measures the variability of returns. In finance, attention has typically focused on variance—σ^2—as the measure of volatility. However, investors knowing F_{t-1} will forecast more accurately with h_t, and this should be the variable used in financial analyses. We will measure volatility by the *conditional* variance of returns.

Note that $\varepsilon_t^2 - h_t$ cannot be forecast and may therefore be considered to be volatility surprise. Whether or not Equation (4) is different from the first term on the right-hand side of Equation (3) is an empirical matter. If volatility is unpredictable, then they will be the same. Our assertion, made at the outset, is that volatility is predictable, hence that h_t is a random variable that depends upon recent information.

If this is so, how far can it take us? Can higher moments be predicted as well? Let the conditional skewness and kurtosis be defined, respectively, as:

Eq. 5

$$s_t = E_{t-1}[(y_t - m_t)/\sqrt{h_t}]^3$$

Eq. 6

$$k_t = E_{t-1}[(y_t - m_t)/\sqrt{h_t}]^4.$$

These potentially depend on past information. It cannot be asserted from theory that these are constants, but there is no empirical evidence yet for time variation in these moments.

Formulating Volatility Equations

The specification problem for analyzing a series y_t can be described by three steps:

1. Specify m_t.
2. Specify h_t.
3. Specify the conditional density of ε_t.

The new part of the problem is Step 2. All previous analyses have faced Steps 1 and 3. We will thus focus on Step 2. For return data, this step is also likely to be the most important. For financial markets, m_t is generally the risk premium, or the expected return. This is frequently zero, at least for high-frequency data. Furthermore, the assumption that the conditional density is normally distributed usually does not appreciably affect the estimates, even if it is false.[3]

This paper focuses on specifying how conditional variances depend upon past information. This is also the equation that will be used to forecast volatility and that can be used to determine time-varying risk premiums. We will not focus on the estimation procedures; these have been dealt with in other papers.[4] It is sufficient to say that maximum-likelihood estimation is generally recommended and used and that the likelihood function typically assumes that the conditional density is Gaussian, so that the logarithmic likelihood of the sample is simply the sum of the individual normal conditional densities.

The simplest specification of the conditional variance equation is the ARCH(p) model, in which the conditional variance is simply a weighted average of past squared forecast errors:

Eq. 7

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2.$$

In the special case where $\omega = 0$ and $\alpha_i = 1/p$, this is simply the sample variance of the previous p returns. This is, in fact, the volatility estimate used by the vast majority of market participants. Different investors choose different values of p, depending on how fast they think volatility is changing. Rarely, however, do they interpret an intercept as a measure of unconditional variance or vary the alphas to reflect the fact that more recent information is probably more valuable than older information. The advantage of the ARCH formulation is that these parameters can be estimated from historical data and used to forecast future patterns in volatility.

Equation (7) can also be written as:

Eq. 8

$$\varepsilon_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + [\varepsilon_t^2 - h_t]$$

where the term in brackets is unforecastable and is therefore considered the innovation in the autoregression for ε^2. This is the source of the name autoregressive conditional heteroskedasticity.

A natural generalization is to allow past conditional variances to enter Equations (7) and (8), giving the generalized ARCH model, or GARCH. The GARCH(p,q) model is:

Eq. 9

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i}$$

Eq. 10

$$\varepsilon_t^2 = \omega + \sum_{i=1}^{p} (\alpha_i + \beta_i) \varepsilon_{t-i}^2 - \sum_{i=1}^{p} \beta_i [\varepsilon_{t-i}^2 - h_{t-i}] + [\varepsilon_t^2 - h_t],$$

where $p \geq q$ is assumed without loss of generality. Note that Equation (10) shows that ε^2 follows an ARMA(p,p). Although the innovations in brackets are serially uncorrelated, they are dependent and heteroskedastic so that estimation using standard Box Jenkins software on ε_t^2 will generally be very inefficient. However, the formulas for forecasting ARMA processes will still apply and can be used for one-step and multistep forecasts of h_t. Even questions of unit roots for long-term forecasting can be addressed in a familiar fashion.[5]

Although the GARCH models provide a rather general class of processes that are linear in ε^2, many other estimators have been proposed. The two most widely used alternative estimators are Schwert's standard deviation model

Eq. 11

$$h_t = \{\omega + \sum_{i=1}^{p} \alpha_i |\varepsilon_{t-i}|\}^2$$

Table I ARCH Models and Sources

AARCH: Augmented ARCH
A. Bera and S. Lee, "On the Formulation of a General Structure for Conditional Heteroskedasticity" (Department of Economics, University of Illinois at Urbana-Champaign, 1989)

AARCH: Asymmetric ARCH
R. F. Engle, "Discussion: Stock Volatility and the Crash of '87," *Review of Financial Studies* 3 (1990), pp. 103–6

GJR
L. Glosten, R. Jaganathan and D. Runkle, "Relationship Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks" (J. L. Kellogg Graduate School, Northwestern University, 1989)

MARCH: Modified ARCH
B. Friedman and D. Laibson, "Economic Implications of Extraordinary Movements in Stock Prices" (Department of Economics, Harvard University, 1989)

MARCH: Multiplicative ARCH
A. Milhoj, "A Multiplicative Parameterization of ARCH Models" (Department of Statistics, University of Copenhagen, 1987)

NARCH: Nonlinear ARCH
R. F. Engle and T. Bollerslev, "Modelling the Persistence of Conditional Variances," *Econometric Review* 5 (1986), pp. 1–50, 81–7; R. F. Engle and V. Ng, "Measuring and Testing the Impact of News on Volatility" (Discussion paper, University of California at San Diego, 1991)
A. Bera and S. Lee, "On the Formulation of a General Structure for Conditional Heteroskedasticity" (Department of Economics, University of Illinois at Urbana-Champaign, 1989)

PNP ARCH: Partially Nonparametric ARCH
R. F. Engle and V. Ng, "Measuring and Testing the Impact of News on Volatility" (Discussion paper, University of California at San Diego, 1991)

QTARCH: Qualitative Threshold ARCH
C. Gourieroux and A. Monfort, "Qualitative Threshold ARCH Models," in Engle and Rothschild, eds., *ARCH Models in Finance*

Table I—continued

SP ARCH: Semiparametric ARCH
R. F. Engle and G. Gonzalez, "Semiparametric ARCH," *Journal of Business and Economic Statistics*, 1991

STARCH: Structural ARCH
A. Harvey, E. Ruiz and E. Sentana, "Structural Unobserved Component ARCH Models," in Engle and Rothschild, eds., *ARCH Models in Finance*

TARCH: Threshold ARCH
J. Zakoian, "Threshold Heteroskedasticity Model" (INSEE, 1991)

and Nelson's EGARCH

Eq. 12

$$\log h_t = \omega + \sum_{i=1}^{p}\beta_i \log h_{t-i} + \sum_{i=1}^{p}\alpha_i|\varepsilon_{t-i}|/h_{t-i}^{1/2} + \sum_{i=1}^{p}\gamma_i\varepsilon_{t-i}/h_{t-i}^{1/2}.$$

Equation (12) allows positive and negative values of ε to have different impacts on volatility. In addition, the use of logarithms means that the parameters can be negative without the variance becoming negative.

Table I lists a host of other models. Many of these have advantages over the original ARCH in fitting particular times series. There is little limit to the range of possibilities; the real issue is which are most useful for modeling and forecasting volatility. Most investigators have found that the GARCH (1,1) is a generally excellent model for a wide range of financial data.[6] Only when special needs arise should an investigator feel compelled to go through the entire menu given in Table I.

Time-Varying Risk Premiums

An asset with high expected risk must offer a high rate of return to induce investors to hold it. In the simplest case, this means that increases in conditional variance should be associated with increases in the conditional mean. Merton derives an equation that relates the expected return on the market linearly to the conditional variance of the market:[7]

Eq. 13

$$m_t = \delta h_t,$$

where δ may be interpreted as the coefficient of relative risk aversion of a representative agent. In this sense, m_t is interpreted as a time-varying risk premium. Many studies have examined such models.[8]

Engle, Lilien and Robins propose the ARCH-M or GARCH-M model, which incorporates an equation like Equation (13) as the specification of the mean in Equation (2).[9] It is assumed that:

Eq. 14

$$y_t = \delta g(h_t) + x_{t-1}\beta + \varepsilon_t;$$
$$V_{t-1}(\varepsilon_t) = h_t,$$

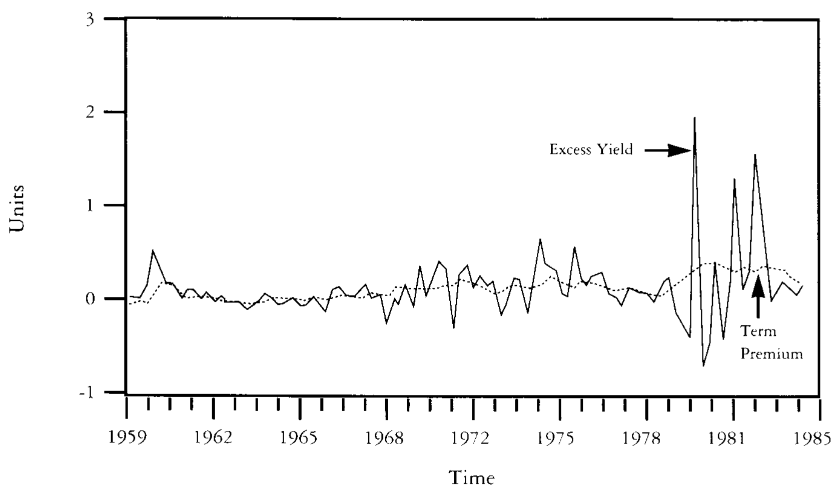
where h_t can be given by any of the available models for variance. The variance of returns this period can be forecast from past information, and the mean can be calculated from past x's and this variance forecast. The introduction of h_t into the mean is thus just another nonlinear function of past information, with the constraint that it responds the same way variance does.

It is well known that the Treasury bill market appears to be inefficient, as the excess return of long-dated bills is predictable. Mankiw and Summers fit the following regression model:[10]

Eq. 15

$$y_t = -0.50 + 2.44(R_t - r_t) + e_t,$$
$$(-1.10)\quad(5.46)$$

Figure B Excess Holding-Period Yield of Six-Month Treasury Bills and Estimated Risk Premiums



where y_t is the quarterly return from holding a six-month Treasury bill for three months and then selling it, financing the purchase by borrowing at the three-month rate. This excess yield requires no net investment and yet seems predictable based on the spread between the six-month rate, R_t, and the three-month rate, r_t, both of which are known at the beginning of the quarter.

Although there have been many explanations for the failure of the Treasury bill market to arbitrage expected profits, Engle, Lilien and Robins propose an explanation based on a time-varying risk premium. Just when there appear to be positive expected profits, the market knows that the risk is especially high, so the profit opportunity remains unexploited. Technically, Equation (15) is misspecified for two reasons. First, it omits a time-varying risk premium; second, it has **heteroskedasticity**, so that the t-statistics do not correctly reveal the significance of the regressors. Engle et al. proceed to estimate the ARCH-M model:

Eq. 16

$$y_t = \underset{(4.38)}{0.355} + \underset{(3.36)}{0.068} \log h_t + e_t$$

$$h_t = \underset{(2.22)}{0.055} + \underset{(5.56)}{0.148} \sum_{i=1}^{p} (5 - i)\varepsilon_{t-i}^2,$$

which shows that the ARCH effect is significant as judged by the t-statistic on the second coefficient in the variance equation, and that the time-varying risk premium is significant in the mean, with a t-ratio of 3.36. When this model is tested for an omitted variable ($R_t - r_t$), using a Lagrange multiplier test, the test is not significant. However, when it is reentered in the equation, which therefore nests both Equations (15) and (16), the coefficient on the spread drops to 0.39 but still has a t-ratio of 2.58. There thus appears to be some evidence of predictable returns beyond the risk premium, but the magnitude is much smaller. Figure B plots the returns and the expected returns from Equation (16).

CAPM and APT

Individual assets cannot in general be priced by their own variance, because only nondiversifiable risk should be rewarded. Thus risk premiums are generally expressed in terms of the asset's covariance with particular reference portfolios. In the Capital Asset Pricing Model (CAPM), the covariance with the market return is the theoretical source of risk, while in Arbitrage Pricing Theory (APT), there may be several sources of risk.

Consider a simple version of the APT, which has only two factors—a market factor, R_{mt}, and a volatility factor, V_t. The return on asset i is thus given by:

Eq. 17

$$r_{it} = \alpha_i + \beta_i R_{mt} + \gamma_i V_t + \varepsilon_t,$$

where β and γ are the **factor loadings**. An agent who dislikes volatility will seek to hedge volatility increases by incorporating in his portfolio assets that have positive γ_i's, since these will increase in value when volatility increases. Several such assets are available. The most familiar are options, particularly when hedged for price-level moves or when a straddle is taken. In addition, many stocks have important option characteristics, and others are assets of firms that thrive on volatility.

Of course, it must be recognized that these high-γ firms are in demand and that they presumably have a somewhat lower expected rate of return. Current research is investigating the strength of this effect and various ways to measure and classify firms.[11]

The assumption that the factor loadings are constant over time is not grounded in economic theory, but in pragmatism. Recognizing that β is generally the ratio of the covariance of an asset with the market to the variance of the market suggests that covariances and betas are possibly forecastable in the same way variances are forecastable. There are thus several reasons to be interested in multivariate ARCH processes that model not only variances, but also covariances.

One can define the conditional mean and conditional variance-covariance matrix for a set of asset returns given by a vector y_t as follows:

Eq. 18

$$E_{t-1}[y_t] = m_t,$$

$$V_{t-1}[y_t] = H_t.$$

If the market share of the assets is given by w_{t-1}, then the market return is simply:

$$R_{mt} = y_t' w_{t-1}$$

and the market variance and covariances with individual security returns are:

Eq. 19

$$V_{t-1}(R_{mt}) = w_{t-1}' H_t w_{t-1},$$

$$Cov_{t-1}(y_t, R_{mt}) = H_t w_{t-1}.$$

Finally, the vector of betas for all assets can be defined as the ratio of the covariance to the variance, or:

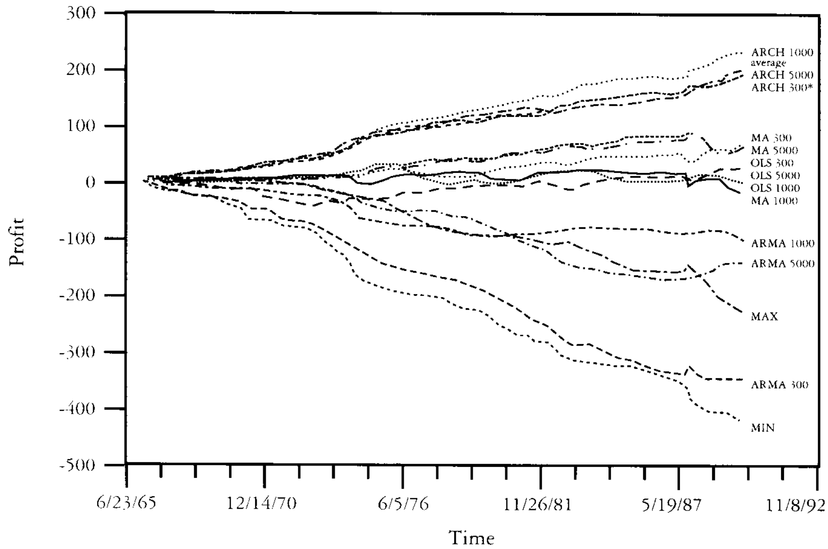
Eq. 20

$$\beta_t = (w_{t-1}' H_t w_{t-1})^{-1} H_t w_{t-1},$$

which will be constant over time only in very special cases. Specification of the covariance matrix in Equation (18) will thus allow calculation of a time-varying beta vector and forecasts of betas.

This model has been estimated by Bollerslev, Engle and Wooldridge for three assets—stocks, bonds and bills—and by McCurdy and Stengos and Gonzalez for pairs of assets, one of which is a market index.[12] Gonzalez examined the beta between individual technology stocks and the market, while McCurdy and Stengos examined the beta between Japanese equity markets and the world equity market. For larger numbers of assets, more parsimonious structures, such as the factor ARCH, may provide a useful simplification.[13]

Figure C Cumulative Annualized Profits from All Trades, June 23, 1966 to December 29, 1989



Options

It is well known that option prices as computed by the Black-Scholes formula depend upon the variance of the underlying asset. In the Black-Scholes framework, this variance is assumed to be constant, hence its estimation is simple. Many practitioners believe that Black-Scholes provides a good approximation, but that it must have up-to-date variance estimates, even possibly the implied volatilities from some other contract or previous trade. The variance from an ARCH model might provide a good estimate for pricing options.

There are more wide-sweeping alternatives to Black-Scholes pricing, which recognize that unless there is an asset perfectly correlated with volatility, there is no longer a riskless dynamic hedge portfolio and there may even be risk premiums. Furthermore, the hedge ratios may differ and will surely change with volatility shocks. Day and Lewis, Lameroux and Lastrapes, and Bartunek and Mustafa find that volatilities implied by options prices do not capture all available information about the future volatility of the underlying asset.[14] Engle and Mustafa compute the entire *implied* distribution of the underlying asset.[15] They ask what GARCH model for the underlying asset is most consistent with the full set of market option prices at a particular time. This implied GARCH model is very similar to the historically estimated GARCH, except around the October 1987 market crash when the option market viewed the volatility shock as less persistent than usual.

Engle, Hong, Kane and Noh carry out a careful evaluation of the value of ARCH models for pricing very short-term options.[16] Their study focuses on discovering the best variance estimate to use in the Black-Scholes formula for a one-day option. They set up a synthetic market for such options, where each agent uses a different procedure to calculate the volatility and therefore has a different advertised price. These agents then buy, sell and hedge options with the expectation that they are making arbitrage profits. At the end of the day, the stock price is revealed and accounts are settled. This is carried out for approximately 5000 days, using NYSE daily data from CRSP.

The authors examine 12 different variance forecasting methods—including two traditional approaches (moving variance estimators and OLS estimators) and newer methods such as ARCH and ARMA estimation of squared residuals. In each case, the estimator uses the most recent 300 days, 1000 days or all prior information in making the forecasts. In addition to the 12 estimators, the study examines the results of three noise traders who do not think for themselves. One uses the average of all volatility forecasts, a second uses the minimum and the third uses the maximum.

In bilateral and all multilateral trading arrangements, the ARCH process proved the most profitable, which indicates that this approach to pricing options is closest to matching the true value of options on the NYSE. The only close competitor is the average, but it does not do well when some of the inferior methods are dropped. Figure C presents the cumulative profit positions of the various approaches over the sample period. It is clear that the best models increase their profits in almost all periods, while the worst lose in almost all periods. This is strong evidence of the short-run forecasting ability of the ARCH models.

Footnotes

1. *R. F. Engle, "Autoregressive Conditional Heteroskedasticity with Estimates of Variance of U.K. Inflation,"* Econometrica *50 (1982), pp. 987–1008.*
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3. *See, for example, R. F. Engle and G. Gonzalez, "Semiparametric ARCH,"* Journal of Business and Economic Statistics *9 (1991), pp. 345–60.*
4. *See Engle, "Autoregressive Conditional Heteroskedasticity,"* op. cit. *and Bollerslev, "Generalized Autoregressive Conditional Heteroskedasticity,"* op. cit.
5. *See R. F. Engle and T. Bollerslev, "Modelling the Persistence of Conditional Variances,"* Econometric Review *5 (1986), pp. 1–50, 81–87.*
6. *See the survey by T. Bollerslev, R. Chou and K. Kroner, "ARCH Modelling in Finance: A Selective Review of the Theory and Empirical Evidence with Suggestions for Future Research," in Engle and Rothschild, eds.,* ARCH Models in Finance, Journal of Econometrics *52(1992), pp. 5–59.*
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8. *See K. French, W. Schwert and R. Stambaugh, "Expected Stock Returns and Volatility,"* Journal of Financial Economics *19 (1986), pp. 3–29, and R. Y. Chou, "Volatility Persistence and Stock Valuations: Some Empirical Evidence Using GARCH,"* Journal of Applied Econometrics *3 (1988), pp. 279–94 for the stock market and R. F. Engle, D. Lilien and R. Robins, "Estimating Time Varying Risk Premia in the Term Structure; The ARCH-M Model,"* Econometrica *55 (1987) for T-bills.*
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11. *This notion of a volatility factor is consistent with the measure of coskewness in A. Kraus and R. Litzenberger, "Skewness Preference and the Valuation of Risk Assets,"* Journal of Finance *31 (1978), pp. 1085–100.*
12. *T. Bollerslev, R. Engle and J. Wooldridge, "A Capital Asset Pricing Model with Time Varying Covariances,"* Journal of Political Economy *96 (1988), pp. 116–31; T. McCurdy and Stengos, "A Comparison of Risk-Premium Forecasts Implied by Parametric vs. Nonparametric Conditional Mean Estimators," in* ARCH Models in Finance, op. cit, *pp. 225–44; and G. Gonzalez (Ph.D. Dissertation, University of California at San Diego, 1991).*
13. *See, for example, R. F. Engle, V. Ng and M. Rothschild, "Asset Pricing with a Factor ARCH Covariance Structure: Empirical Estimates for Treasury Bills,"* Journal of Econometrics *45 (1990), pp. 213–39; W. Schwert and P. Seguin, "Heteroskedasticity in Stock Returns,"* Journal of Finance *45 (1991), pp. 1129–56; and V. Ng, R. Engle and M. Rothschild, "A Dynamic Multi-Factor Model of Stock Returns," in Engle and Rothschild, eds.,* ARCH Models in Finance, op. cit., *pp. 245–66.*
14. *T. Day and C. Lewis, "Stock Market Volatility and the Information Content of Stock Index Options," in Engle and Rothschild, eds.,* ARCH Models in Finance, op. cit.; *C. Lamoureux and W. Lastrapes, "Forecasting Stock Return Variance: Toward an Understanding of Stochastic Implied Volatilities" (1990); and K. Bartunek and C. Mustafa, "Forecasting the Variance of the Underlying Asset of an Option: A Comparison of Conditional and Unconditional Forecasts" (Louisiana State University, 1991).*
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