Modigliani-Miller Theorem

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Abstract

The Modigliani-Miller Theorem states that the value of a firm is invariant with respect to its leverage policy in an arbitrage-free market when there is no corporate income tax and no bankruptcy cost: whether the firm is financed through debt or equity, its value remains the same. We present various forms of this result and contrat it with empirical evidence on capital structure of firms.

The Modigliani-Miller Theorem states that the value of a firm is invariant with respect to its leverage policy in an arbitrage-free market when there is no corporate income tax and no bankruptcy cost. Modigliani and Miller (1958) proved this "invariance" result, using for the first time, a no-arbitrage condition. To present the M-M theorem, we shall consider a simple levered firm in which there are two basic claimants on the firm's income :

- *debt holders*, whose security allows them to claim the coupon C at each time t as long as default is not declared.
- *equity holders* (the owners of the firm), whose security allows them, once debt holders have been paid, to claim the *residual cash-flow* (if positive) as dividends.

Debt holders are paid *before* equity holders, but their claim does not allow them to receive more than the coupon, no matter what the net result is. On the contrary, dividends received by equity holders may be very high when the net result is very high but can also be very low (and even null) when the net result is very low. Equity holders are frequently called the "residual claimant" since they own the *residual income* (i.e., the net result) of the firm, that is, what remains when employees, debt holders, and government has been paid.

Let E^l and B be the market value of equity and debt respectvely of a given firm at some point in time. To become the owner of this (levered) firm, an investor must buy the equity E^l, but also the debt of the firm, so that the total payment is equal to $V^l = E^l + B$, where V^l is called the *value of the firm*. It is thus tempting to think that if we increase B, the value of the firm V^l will also increase. Finding conditions under which debt affects, or not, the value of the firm is the central question of Modigliani-Miller analysis: given a *fixed* investment policy, under what condition(s) debt does not affect the value of the firm ?

1 Basic statement of the theorem

Consider a firm who has invested in some productive asset which allows her to generate an operating income (i.e., the earning before interests and taxes (EBIT)) X_t at each time time t. Roughly speaking, X_t is equal to the value of the sales minus production costs. When positive, X_t is used to pay debt holders (coupon), government (tax), and equity holders (dividends).

Unlevered firm. In this case, the value of the firm at time t is denoted v_t, and is simply equal to the net result, that is:

$$v_t = X_t(1 - T_a) \tag{1}$$

where T_a is the corporate tax income.

Let C be the coupon of the corporate debt, T be the maturity of the debt. In what follows, we shall only consider the case of a *perpetual debt* so that T is infinite, but we shall also assume that each firm distributes all the residual income (or cash flow) as dividends to equity holders. This thus implies that the firm does not have any cash reserve that it can use when the EBIT is not enough to pay the coupon C. Let v_t^l be the value of the levered firm at a given time t. We shall now compute v_t^l under various scenarios. Let

Levered firm, no default risk. We say that a perpetual corporate debt is *risk-free* if the coupon C paid at each time t to debt holders with probability one. Since the debt is risk-free, the net result $(X_t - C)(1 - T_a)$ is positive with probability one[1]. The value of the levered firm at time t is thus equal to:

$$v_t^l = C + (X_t - C)(1 - T_a) = X_t(1 - T_a) + T_a C \quad (2)$$
$$= v_t + T_a C \quad (3)$$

The value of the levered firm is equal to the value of the unlevered firm v_t, plus the value of the tax-shield $T_a C$, which comes from the fact that interests expenses are tax-deductible. In the particular case in which $T_a = 0$, we thus have the following equality:

$$v_t^l = v_t = X_t \quad (4)$$

When there is no corporate income tax, the value of the levered firm (at time t) is equal to the value of the unlevered one (at time t), i.e., the value of the firm is invariant with respect to C as long as the coupon implies no default risk. This is the Modigliani-Miller theorem without default risk.

Levered firm, default risk. We say that the corporate bond is risky if there is a positive probability that the coupon will not be paid to debt holders at some time t. Since all the residual income is distributed to equity holders as dividends, there are no available retained earnings so that default is declared at the first time for which $X_t < C$. Note importantly that in a default[2] situation:

- debt holders have an unconditional right to be paid before equity holders.
- the firm is not subject to corporate taxation.
- due to *limited liability* of equity holders, debt holders cannot force them to pay the difference $C - X_t$.
- the bankruptcy cost (i.e., lawyer cost, administrative costs, etc...) is a fraction γ of X_t.

It follows thus that

$$v_t^l = \min\{X_t, C\}(1 - \gamma \mathbf{1}_{X_t < C}) + \max\{(X_t - C)(1 - T_a \mathbf{1}_{X_t \geq C}); 0\} \quad (5)$$

where $\mathbf{1}_{X_t \geq C}$ and $\mathbf{1}_{X_t < C}$ are indicator functions[3]. When there is no default, the value of the firm is given by equation (3). However, when there is default, the value of the firm is equal to $X_t(1 - \gamma)$. In the particular case in which $\gamma = 0$ and $T_a = 0$, it follows that:

$$v_t^l = \min\{X_t, C\} + \max\{(X_t - C); 0\} = v_t = X_t \quad (6)$$

When there is no bankruptcy cost and no corporate income tax, the value of the levered firm is equal to the value of the unlevered one, i.e., the value of the firm is invariant with respect to the coupon. This is the Modigliani-Miller theorem with default risk.

[1] This might not be the case if for example equity holders issue new equity to pay $C - X_t$ to bondholders when $X_t < C$.

[2] We make here no difference between default and liquidation.

[3] Recall that the indicator function of an event A is such that $\mathbf{1}_A = 1$ if $\omega \in A$ and 0 otherwise.

2 Classical presentation

In their original work, M-M (1958) make the following assumptions[4].

1. Capital markets are perfectly competitive.

2. Individuals and firms can borrow and lend at the risk-free rate r.

3. All firms are assumed to be in the same *class risk*.

Assumption 1 implies that market investors have full (and symmetric) information concerning the return of the firm, that there are no transaction costs and no restriction on asset trade, i.e., long and short positions are possible. Assumption 2 means that when firms or households borrow, they are not subject to default risk so that they can borrow at the risk-free rate. Assumption 3 means that the stream of EBIT is the same for all firms in the same class risk; if two firms, one levered and one unlevered belong to the same class risk, then, they differ only with leverage.

In their original work, M-M 1958 consider a discrete time model in which the family of random variable $\{X_t\}_{t\geq 1}$ is identically and independently distributed. Let $\mathbb{E}(X_t) = \bar{X}$ be the expectation of X_t under the probability measure $\mathbb{P}$. In the present framework, the corporate debt is riskless if $\mathbb{P}(X_t > C) = 1$.

2.1 Valuation principle and the cost of capital

In the "classical" approach, the valuation principle is fairly simple: the present value of a stream of *risky* cash flow is equal to the *expected* sum over time of the *discounted* value of the cash flow. The expectation is done under the probability measure $\mathbb{P}$, and the discount factor, also called *risk adjusted rate of return* or more simply the *cost of capital*, incorporates a *risk premium*. Since in general investors are risk averse, the riskier is the cash flow, the higher is the cost of capital. Let r be the cost of capital of a riskless cash flow and $k_u > r$ be the cost of capital of a risky cash flow.

Using the above principle, the value of the unlevered firm is equal to the present value of the flow of v_t, that is:

$$V = E = \mathbb{E}\sum_{t=1}^{\infty} \frac{v_t}{(1+k_u)^t} = \frac{\bar{X}(1-T_a)}{k_u} \tag{7}$$

where V denotes the value of the unlevered firm, which is also equal to the value of equity E. To determine now the value of the levered firm, we have to compute the expected discounted value of the flow of v_t^l. Using equation (3), we just have to compute the value of the stream of tax shield. By assumption, the corporate debt is a perpetual risk-free one, so that the relevant cost of capital for the tax shield is r. It thus follows that:

$$V^l = \mathbb{E}\sum_{t=1}^{\infty} \frac{X_t(1-T_a)}{(1+k_u)^t} + \sum_{t=1}^{\infty} \frac{T_a\, C}{(1+r)^t} = V + T_a\, \frac{C}{r} = V + T_a B \tag{8}$$

where $B = \frac{C}{r}$ is the value of the perpetual corporate risk-free debt, and $T_a B$ the value of the perpetual tax shield. The value of the levered firm is thus equal to the value of the unlevered

[4] See Stiglitz (1988) for a lucid discussion of their assumptions.

firm plus the value of the tax shield. We can now state the famous invariance result when $T_a = 0$. Let $\frac{B}{E^l}$ be the *debt-equity ratio* where $E^l = V^l - B$ is the value of equity of the levered firm.

Modigliani-Miller theorem with risk-free debt. In addition to the above assumptions, suppose that there is no arbitrage opportunity and no corporate income tax (i.e., $T_a = 0$), then, $V = V^l$, i.e., the value of the firm is invariant with respect to the debt-equity ratio.

This result says that when there are no taxes and no default risk, the value of the firm is *invariant* with respect to the "financing policy", which means that issuing only equity, or issuing a mix of debt and equity does not affect the value of the firm. Fundamentally, the value of the firm is determined by the stream of EBIT, that is, by investment policy; debt affects only the *split* of the value of the firm between debt holders and equity holders. As a consequence, different debt-equity ratio leads to different split of the total value but has no impact on this total value.

From a mathematical point of view, the proof is trivial since when $T_a = 0$, we immediately get that $V = V^l$. Actually, Modigliani and Miller become very famous for the result *per se*, but also because they offered for the first time a *no-arbitrage principle*[5] for security valuation: they showed that if $V^l > V$ when $T_a = 0$, then, there is an arbitrage opportunity.

While the proof is rather simple, it highlights their assumptions. Consider two firms that are in the same class risk. The first is unlevered while the second is levered. Since $T_a = 0$, the value of equity of the levered firm is equal to $E^l = V - C/r$ while the value of equity of the unlevered firm is $V = E$. Consider now an equity holder who owns $\kappa\%$ of the shares of the levered firm. At each time t, his claim D_t (i.e., dividend) is equal to:

$$D_t = \kappa(X_t - C) \tag{9}$$

Suppose now that he sells his part of the levered firm for an amount κE^l and borrow an amount equal to $\kappa(C/r)$. With that money, he buys a fraction $\dfrac{\kappa(E^l + (C/r))}{V}$ of the shares of the unlevered firm. Actually, the equity holder borrows an amount $\kappa C/r$ to *replicate* the leverage of the firm, and this has been called a *homemade leverage.* Note importantly that by assumption 2, the investor borrows at the risk-free rate. He thus holds a portfolio in which he will perceive the dividends (in proportion of his part) of the unlevered firm at each time t but will also have to pay the interests expense equal to κC. The return of his portfolio is thus equal to $Y_t = \frac{\kappa(E^l + C/r)}{V} X_t - \kappa C$. By definition, $V^l = E^l + B$ so that Y_t is equal at each time t to:

$$Y_t = \kappa\left(\frac{V^l}{V} X_t - C\right) \qquad t \geq 1 \tag{10}$$

Since $V^l > V$ by assumption, it thus follows that $Y_t > D_t$ with probability one, which is an arbitrage opportunity. If we rule out such an arbitrage opportunity, then, $V^l = V$.

As Duffie (1992) remarks in his review on Modigliani-Miller theorem, the result follows because equity holders can undertake the same financial transaction as the firm in exactly the

[5] Their no-arbitrage proof was indeed mis-understood. See Heins and Sprenkle (1969) and the illuminating reply by Modigliani and Miller (1969).

same condition. If an equity holder is considered as defaultable so that he must borrow at a rate higher than the risk-free rate, then, he cannot offset the financial decision of the firm at the same price.

2.2 The cost of equity and the debt-equity ratio

From equation (7), when $T_a = 0$, the cost of capital $k_u = \bar{X}/E$ is the (expected) rate of return on equity. When the firm is levered, with $T_a = 0$, the (expected) rate of return of the levered firm k_e that we call *cost of equity* is equal to

$$k_e = \frac{\bar{X} - C}{E^l} \tag{11}$$

Since the firms are in the same class risk, the mean EBIT $\bar{X}$ of the two firms are equal. As a consequence, by writing equation (11) as $\bar{X} = E^l k_e + rB$ (recall that $C = rB$), since $\bar{X} = k_u E$, it thus follows that

$$E^l k_e + rB = k_u E \tag{12}$$

By Modigliani-Miller theorem, we know that $V = V^l \equiv E^l + B$. Since $V = E$, from equation (12), we easily obtain that

$$k_e = k_u + (k_u - r)\frac{B}{E^l} \tag{13}$$

Equation (13) means that *ceteris paribus*, the higher is the debt-equity ratio, the higher is the cost of equity of the levered firm. More precisely, the cost of equity is a *linear* increasing function of the debt-equity ratio, and is sometimes thought as a pricing formula where $(k_u - r)\frac{B}{E^l}$ is the premium for the *financial risk*. See Taggart (1991) for a presentation of the various existing formulae of the cost of capital.

2.3 Contrast with empirical evidence

In contrast to the Modigliani-Miller theorem, the value of firms is actually influenced by their capital structure: the debt-equity ratio affects the value of the firm. To explain the observed debt-equity ratio, a large part of the recent theory of capital structure has been to explore the role of the various conflicts, inexistent in the Modigliani Miller analysis, that arise between managers, equity holders and debt holders. Informational asymmetries play an important role. The paper of Harris and Raviv (1991) and the books of Milgrom and Roberts (1992) or Hillier Grinblatt Titman (2008) contain a very readable presentation of many aspects of the recent theory of capital structure.

3 Modern presentation

We now present the Modigliani-Miller theorem in a continuous time setting assuming that markets are arbitrage free and complete[6] so that there exists a *unique pricing measure* (also

[6] In a section entitled "Arrow-Debreu securities", Stiglitz (1969) shows the M-M theorem in a (static) complete markets setting. For a treatment of the incomplete markets case, see De Marzo (1988) or Gottardi (1995).

called risk-neutral measure) denoted $\mathbb{Q}$. More precisely, we assume that under the unique risk-neutral measure $\mathbb{Q}$, the stochastic process $X = (X_t)_{t>0}$ is given by:

$$X_t = x\ e^{(\mu - 0.5\sigma^2)t + \sigma W_t} \Longleftrightarrow \frac{dX_t}{X_t} = \mu dt + \sigma dW_t \tag{14}$$

where μ is the risk-neutral drift, σ the volatility, W_t a standard brownian motion under $\mathbb{Q}$, and x is the value of the EBIT at time $t = 0$. In what follows, the expectation is taken under the pricing measure $\mathbb{Q}$.

3.1 Valuation principle

In the "modern" approach, one also value a stream of risky cash flow as the *expected* sum over time of the *discounted* value of the cash flow. The discount rate is now always equal to the risk free rate r, whether the cash flow is risky or not, and, to prevent arbitrage[7], the pricing measure $\mathbb{Q}$ is such that the *overall expected rate of return* of the unlevered equity, i.e., expected capital gain plus dividend yield, is equal to the risk free rate r.

The value of the unlevered firm seen from $t = 0$, which is also the value of equity, is equal to the the expected sum of discounted v_t (see equation (1)) over time, that is:

$$V(x) \equiv V = \mathbb{E} \int_0^\infty e^{-rt} v_t dt = \mathbb{E} \int_0^\infty e^{-rt} X_t (1 - T_a) dt \tag{15}$$

$$= \frac{x(1 - T_a)}{r - \mu} \qquad \mu < r \tag{16}$$

Seen from a given time $t > 0$, the value of the unlevered firm (or equity) is equal to:

$$V_t = \frac{X_t(1 - T_a)}{r - \mu} \tag{17}$$

We assume, as previously, that equity is a *traded asset*, whose price is V_t at time t. Since all the residual cash-flow is distributed to equity holders as dividends, the dividend yield is equal to $\frac{X_t(1-T_a)}{V_t} = r - \mu$, which means that $\mu < r$. As a consequence, the overall expected rate of return from holding the equity of the unlevered firm is equal to r, that is, the expected capital gain μ plus the dividend yield $r - \mu$.

3.2 Default risk

The existence of default risk complicates the analysis of the preceding section since the flow of v_t^l may be stopped at a random time τ. Moreover, if default occurs at time τ, a fraction γV is lost due to bankruptcy costs. As a consequence, the value of the levered firm is equal to V, plus the value of the tax-shield TS, minus the value of the bankruptcy costs BC. The value of the levered firm V^l at time $t = 0$ is thus equal to:

$$V^l = V + TS - BC \tag{18}$$

To compute the value (at time $t = 0$) of the tax-shield TS and of the bankruptcy costs BC, we now have to precise the default declaration mechanism. It will be assumed that default is

[7] See e.g, Kerry Back (2005), "A Course in Derivatives Securities", Springer-Verlag, p 42.

triggered at the first time $t > 0$ for which $V_t \leq V_B$, with $V_B < C/r$. Let τ denote the default time. Since (almost all) the sample paths of the stochastic process V_t is continuous, default occurs at the first time for which $V_t = V_B$. It follows that:

$$\tau = \inf\{t \in \mathbb{R}^+ : \; V_t = V_B\} \tag{19}$$

It can be shown[8] that, for a perpetual corporate debt, the risk-neutral default probability is equal to:

$$\left(\frac{V_B}{V}\right)^{2(\mu - 0.5\sigma^2)/\sigma^2} \tag{20}$$

To determine TS and BC, we follow Godlstein et al (2001) and shall first compute the present value of a contingent claim that pays 1 euro if the firm defaults. Seen from $t = 0$, if the default time τ were to be known, its value would be equal to $e^{-r\tau}$. Since $\tau > 0$ is not known, the value of such a contingent claim is equal to its expected value. It can be shown[9] that

$$\lim_{T\to\infty} \mathbb{E}\left(e^{-r\tau}\mathbf{1}_{\tau<T}\right) = \left(\frac{V_B}{V}\right)^{\xi} \tag{21}$$

where $\xi = \frac{1}{\sigma^2}\left[(\mu - 0.5\sigma^2) + \sqrt{(\mu - 0.5\sigma^2)^2 + 2r\sigma^2}\right] > 0$, and $\mathbf{1}_{\tau<T}$ the indicator function of the event $\tau < T$ (rcall that T is the maturity ofd the debt).

Consider now the value of the bankruptcy costs. Seen from $t = 0$, when default occurs at time $\tau > 0$, their value is $e^{-r\tau}\gamma V_B$ while if default never occurs, their value is zero. Using equation (21), it thus follows that the value of the bankruptcy costs are equal to:

$$BC = \lim_{T\to\infty} \mathbb{E}\left(\gamma V_B e^{-r\tau}\mathbf{1}_{\tau<T}\right) = \gamma V_B \left(\frac{V_B}{V}\right)^{\xi} \tag{22}$$

In the same way, since the tax-shield is positive as long as default is not declared, its value is equal to:

$$TS = \left(1 - \left(\frac{V_B}{V}\right)^{\xi}\right) T_a(C/r) \tag{23}$$

The value of the levered firm, defined by equation (18), is thus equal to :

$$V^l = V + \left(1 - \left(\frac{V_B}{V}\right)^{\xi}\right) T_a(C/r) - \left(\frac{V_B}{V}\right)^{\xi} \gamma V_B \tag{24}$$

Note that when V (or x) tends to infinity, equation (24) reduces to equation (8) since default risk vanishes. When $T_a = 0$ and $\gamma = 0$ in equation (24), it thus follows that $V^l = V$.

Modigliani-Miller theorem with risky debt. Assume that markets are arbitrage free and complete and that there are no corporate income tax (i.e., $T_a = 0$) and no bankruptcy

[8]See e.g., Bielecki T, and Rutkowski M, (2001), "*Credit Risk: Modeling, Valuation and Hedging*", Springer-Verlag, p. 68.

[9]See Bielecki T, and Rutkowski M, (2001), p 81 for a probabilistic proof or Goldstein et al (2001) p 491 for an alternative proof.

costs (i.e., $\gamma = 0$), then $V = V^l$, i.e., the value of the firm is invariant with respect to the debt-equity ratio.

Default risk does not invalidate the Modigliani-Miller theorem as long as there are no taxes and no bankruptcy costs. When there are both tax and bankruptcy costs, the debt-equity ratio that maximizes the value of the firm is an optimal trade-off between the value of the tax shield and the value of the bankruptcy costs[10]. However, it is not only assumed that there is no cash reserve (due to the particular dividend policy), but also that equity holders can *costlessly* issue new shares to avoid default if they want to do so.

Assume now that when some fraction of the net income (when positive) is not distributed to equity holders as dividends, it goes on a cash reserve that capitalizes at a rate equal to $r - \lambda > 0$. The parameter λ reflects inefficiencies of managerial expenses (agency costs) so that some fraction of the cash can be considered as lost. When there are both issuance and agency costs, it has been shown that the *optimal dividend policy*[11] is an optimal trade-off between these two costs.

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